STATICS AND DYNAMICS

FOR

ENGINEERING STUDENTS.

BY

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MECHANICS OF ENGINEERING.

PRELIMINARY CHAPTER.

1. **Mechanics** treats of the mutual actions and relative motions of material bodies, solid, liquid, and gaseous; and by *Mechanics of Engineering* is meant a presentment of those principles of pure mechanics, and their applications, which are of special service in engineering problems.

2. **Kind. of Quantity.**—Mechanics involves the following fundamental kinds of quantity: **Space,** of one, two, or three dimensions, i.e., length, surface, or volume, respectively; **time,** which needs no definition here; **force** and **mass,** as defined below; and **abstract numbers,** whose values are independent of arbitrary units, as, for example, a ratio.

3. **Force.**—A force is one of a pair of equal, opposite, and simultaneous actions between two bodies, by which the state of their motions is altered or a change of form in the bodies themselves is effected. Pressure, attraction, repulsion, and traction are instances in point. Muscular sensation conveys the idea of force, while a spring-balance gives an absolute measure of it, a beam-balance only a relative measure. In accordance with Newton's third law of motion, that action and reaction are equal, opposite, and simultaneous, forces always occur in pairs; thus, if a pressure of 40 lbs. exists ' —een bodies A and B, if A is considered by itself (i.e., apart from all other bodies whose actions upon it a forces, among these forces will be one of 40 lbs B toward A. Similarly, if B is under consid

of 40 lbs. directed from A toward B takes its place among the forces acting on B. This is the interpretation of Newton's third law.

In conceiving of a force as applied at a certain point of a body it is useful to imagine one end of an imponderable spiral spring in a state of compression (or tension) as attached at the given point, the axis of the spring having the given direction of the force.

4. **Mass** is the quantity of matter in a body. The masses of several bodies being proportional to their weights at the same locality on the earth's surface, in physics the weight is taken as the mass, but in practical engineering another mode is used for measuring it (as explained in a subsequent chapter), viz.: the mass of a body is equal to its weight divided by the acceleration of gravity in the locality where the weight is taken, or, symbolically, $M = G \div g$. This quotient is a constant quantity, as it should be, since the mass of a body is invariable wherever the body be carried.

5. **Derived Quantities.**—All kinds of quantity besides the fundamental ones just mentioned are compounds of the latter, formed by multiplication or division, such as velocity, acceleration, momentum, work, energy, moment, power, and force-distribution. Some of these are merely names given for convenience to certain combinations of factors which come together not in dealing with first principles, but as a result of common algebraic transformations.

6. **Homogeneous Equations** are those of such a form that they are true for any arbitrary system of units, and in which all terms combined by algebraic addition are of the same kind. Thus, the equation $s = \frac{gt^2}{2}$ (in which $g =$ the acceleration of gravity and t the time of vertical fall of a body in vacuo, from rest) will give the distance fallen through, s whatever units be adopted for measuring time and distance. But if for

g we write the numerical value 32.2, which it assumes when time is measured in seconds and distance in feet, the equation $s = 16.1t^2$ is true for those units alone, and the equation is not of homogeneous form. Algebraic combination of homogeneous equations should always produce homogeneous equations; if not, some error has been made in the algebraic work. If any equation derived or proposed for practical use is not homogeneous, an explicit statement should be made in the context as to the proper units to be employed.

7. Heaviness.—By heaviness of a substance is meant the weight of a cubic unit of the substance. E.g. the heaviness of fresh water is 62.5, in case the unit of force is the pound, and the foot the unit of space; i.e., a cubic foot of fresh water weighs 62.5 lbs. In case the substance is not uniform in composition, the heaviness varies from point to point. If the weight of a homogeneous body be denoted by G, its volume by V, and the heaviness of its substance by γ, then $G = V\gamma$.

WEIGHT IN POUNDS OF A CUBIC FOOT (i.e., THE HEAVINESS) OF VARIOUS MATERIALS.

Anthracite, solid	100	Masonry, dry rubble	138
" broken	57	" dressed granite or limestone	165
Brick, common hard	125	Mortar	100
" soft	100	Petroleum	55
Brick-work, common	112	Snow	7
Concrete	125	" wet	15 to 50
Earth, loose	72	Steel	490
" as mud	102	Timber	25 to 60
Granite	164 to 172	Water, fresh	62.5
Ice	58	" sea	64.0
Iron, cast	450		
" wrought	480		

8. Specific Gravity is the ratio of the heaviness of a material to that of water, and is therefore an abstract number.

9. A Material Point is a solid body, or small particle, whose dimensions are practically nothing, compared with its range of motion.

10. A Rigid Body is a solid, whose distortion or change of form under any system of forces to be brought upon it in practice is, for certain purposes, insensible.

11. Equilibrium.—When a system of forces applied to a body produces the same effect as if no force acted, so far as the *state of motion* of the body is concerned, they are said to be balanced, or to be in equilibrium.

12. Division of the Subject.—*Statics* will treat of bodies at rest, i.e., of balanced forces or equilibrium; *dynamics*, of bodies in motion; *strength of materials* will treat of the effect of forces in distorting bodies; *hydraulics*, of the mechanics of liquids; *pneumatics*, of the mechanics of gases.

13. Parallelogram of Forces.—Duchayla's Proof. To fully determine a force we must have given its amount, its direction, and its point of application in the body. It is generally denoted in diagrams by an arrow. It is a matter of experience that besides the point of application already spoken of any other may be chosen in the line of action of the force. This is called the transmissibility of force; i.e., so far as the *state of motion* of the body is concerned, a force may be applied anywhere in its line of action.

The **Resultant** of two forces (called its components) applied at a point of a body is a single force applied at the same point, which will replace them. To prove that this resultant is given in amount and position by the diagonal of the parallelogram formed on the two given forces (conceived as laid off to some scale, so many pounds to the inch, say), Duchayla's method requires four postulates, viz.: (1) the resultant of two forces must lie in the same plane with them; (2) the resultant of two equal forces must bisect the angle between them; (3) if one of the two forces be increased, the angle between the other force and the resultant will be greater than before; and (4) the trans missibility of force, already mentioned. Granting these, we proceed as follows (Fig. 1): Given the two forces P and $Q =$

$P' + P''$ (P' and P'' being each equal to P, so that $Q = 2P$), applied at O. Transmit P'' to A. Draw the parallelograms OB and AD; OD will also be a parallelogram. By postulate (2), since OB is a rhombus, P and P' at O may be replaced by a single force R' acting through B. Transmit R' to B and replace it by P and P'. Transmit P from B to A, P' from B to D. Similarly P and P'', at A, may be replaced by a single force R'' passing through D; transmit it there and resolve it into P and P''. P' is already at D. Hence P and $P' + P''$, acting at D, are equivalent to P and $P' + P''$ acting at O, in their respective directions. Therefore the resultant of P and $P' + P''$ must lie in the line OD, the diagonal of the parallelogram formed on P and $Q = 2P$ at O. Similarly

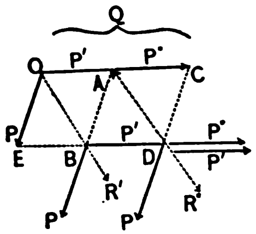


FIG. 1.

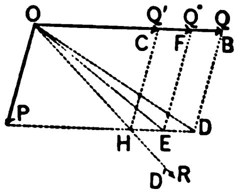


FIG. 2.

this may be proved (that the diagonal gives the *direction* of the resultant) for any two forces P and mP; and for any two forces nP and mP, m and n being any two whole numbers, i.e., for any two commensurable forces. When the forces are incommensurable (Fig. 2), P and Q being the given forces, we may use a *reductio ad absurdum*, thus: Form the parallelogram OD on P and Q applied at O. Suppose for an instant that R the resultant of P and Q does not follow the diagonal OD, but some other direction, as OD'. Note the intersection H, and draw HC parallel to DB. Divide P into equal parts, each less than HD; then in laying off parts equal to these from O along OB, a point of division will come at some point F between C and B. Complete the parallelogram $OFEG$. The force $Q'' = OF$ is commensurable with P, and hence their

resultant acts along OE. Now Q is greater than Q'', while R makes a less angle with P than OE, which is contrary to postulate (3); therefore R cannot lie outside of the line OD. Q. E. D.

It still remains to prove that the resultant is represented in *amount*, as well as position, by the diagonal. OD (Fig. 3) is the direction of R the resultant of P and Q; required its amount. If R' be a force equal and opposite to R it will balance P and Q; i.e, the resultant of R' and P must lie in the line QO prolonged (besides being equal to Q). We can therefore determine R' by drawing BA parallel to DO to intersect QO prolonged in A; and then complete the parallelogram on BA and BO. Since $OFAB$ is a parallelogram R' must $=\overline{BA}$, and since $OABD$ is a parallelogram $\overline{BA}=\overline{OD}$; therefore $R'=\overline{OD}$ and also $R=\overline{OD}$. Q. E. D.

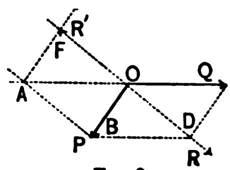


FIG. 3.

Corollary.—The resultant of three forces applied at the same point is the diagonal of the parallelopiped formed on the three forces.

14. Concurrent forces are those whose lines of action intersect in a common point, while **non-concurrent** forces are those which do not so intersect; results obtained for a system of concurrent forces are really derivable, as particular cases, from those pertaining to a system of non-concurrent forces.

15. Resultant.—A single force, the action of which, as regards the *state of motion* of the body acted on, is equivalent to that of a number of forces forming a system, is said to be the **Resultant** of that system, and may replace the system; and conversely a force which is equal and opposite to the resultant of a system will balance that system, or, in other words, when it is combined with that system there will result a new system in equilibrium.

In general, as will be seen, a given system of forces can al-

ways be replaced by two single forces, but these two can be combined into a single resultant only in particular cases.

15a. Equivalent Systems are those which may be replaced by the same set of two single forces—or, in other words, those which have the same effect, as to state of motion, upon the given body.

PART I.—STATICS.

CHAPTER I.

STATICS OF A MATERIAL POINT.

16. Composition of Concurrent Forces.—A system of forces acting on a material point is necessarily composed of concurrent forces.

CASE I.—All the forces in **One Plane.** Let O be the material point, the common point of application of all the forces; P_1, P_2, etc., the given forces, making angles α_1, α_2, etc., with the axis X. By the parallelogram of forces P_1 may be resolved into and replaced by its components, $P_1 \cos \alpha$ acting along X, and $P_1 \sin \alpha$ along Y. Similarly all the remaining forces may be replaced by their X and Y components. We have now a new system, the equivalent of that first given, consisting of a set of X forces, having the same line of application (axis X), and a set of Y forces, all acting in the line Y. The resultant of the X forces being their algebraic sum (denoted by ΣX) (since they have the same line of application) we have

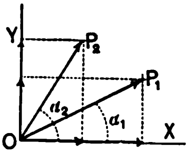


FIG. 4.

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \text{etc.} = \Sigma(P \cos \alpha),$$

and similarly

$$\Sigma Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \text{etc.} = \Sigma(P \sin \alpha).$$

These two forces, ΣX and ΣY, may be combined by the parallelogram of forces, giving $R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ as the single resultant of the whole system, and its direction is determined by the angle α; thus, $\tan \alpha = \frac{\Sigma Y}{\Sigma X}$; see Fig. 5. For equilibrium to exist, R must $= 0$, which requires, *separately*,

$\Sigma X = 0$, and $\Sigma Y = 0$ (for the two squares $(\Sigma X)^2$ and $(\Sigma Y)^2$ can neither of them be negative quantities).

Case II.—The forces having any directions in space, but all applied at O, the material point. Let P_1, P_2, etc., be the given forces, P_1 making the angles α_1, β_1, and γ_1, respectively, with three arbitrary axes, X, Y, and Z (Fig. 6), at right angles to each other and intersecting at O, the origin. Similarly let α_2, β_2, γ_2, be the angles made by P_2 with these axes, and so on for all the forces. By the parallelopiped of velocities, P_1 may be replaced by its components,

$X_1 = P_1 \cos \alpha_1$, $Y = P_1 \cos \beta_1$, and $Z_1 = P_1 \cos \gamma_1$; and

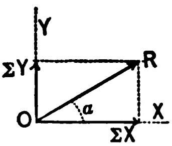


Fig. 5.

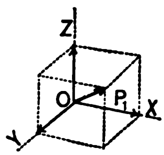


Fig. 6.

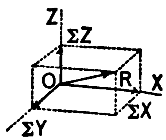


Fig. 7.

similarly for all the forces, so that the entire system is now replaced by the three forces,

$$\Sigma X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \text{etc};$$
$$\Sigma Y = P_1 \cos \beta_1 + P_2 \cos \beta_2 + \text{etc};$$
$$\Sigma Z = P_1 \cos \gamma_1 + P_2 \cos \gamma_2 + \text{etc};$$

and finally by the single resultant

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

Therefore, for equilibrium we must have **separately**,

$$\Sigma X = 0, \ \Sigma Y = 0, \text{ and } \Sigma Z = 0.$$

R's position may be determined by its direction cosines, viz.,

$$\cos \alpha = \frac{\Sigma X}{R}; \ \cos \beta = \frac{\Sigma Y}{R}; \ \cos \gamma = \frac{\Sigma Z}{R}.$$

17. Conditions of Equilibrium.—Evidently, i· ·ith a system of concurrent forces, it would b·

replace any two of the forces by their resultant (diagonal formed on them), then to combine this resultant with a third force, and so on until all the forces had been combined, the last resultant being the resultant of the whole system. The foregoing treatment, however, is useful in showing that for equilibrium of concurrent forces in a plane only two conditions are necessary, viz., $\Sigma X = 0$ and $\Sigma Y = 0$; while in space there are three, $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma Z = 0$. In Case I., then, we have conditions enough for determining two unknown quantities; in Case II., three.

18. Problems involving equilibrium of concurrent forces. (A rigid body in equilibrium under no more than three forces may be treated as a material point, since the (two or) three forces are necessarily concurrent.)

PROBLEM 1.—A body weighing G lbs. rests on a horizontal table: required the pressure between it and the table. Fig. 8. Consider the body free, i.e., conceive all other bodies removed (the table in this instance), being replaced by the forces which they exert on the first body. Taking the axis Y vertical and positive upward, and not assuming in advance either the amount or direction of N, the pressure of the table against the body, but knowing that G, the action of the earth on the body, is vertical and downward, we have here a system of concurrent forces in equilibrium, in which the X and Y components of G are known (being 0 and $-G$ respectively), while those, N_X and N_Y, of N are unknown. Putting $\Sigma X = 0$, we have $N_X + 0 = 0$; i.e., N has no horizontal component, $\therefore N$ is vertical. Putting $\Sigma Y = 0$, we have $N_Y - G = 0$, $\therefore N_Y = +G$; or the vertical component of N, i.e., N itself, is positive (upward in this case), and is numerically equal to G.

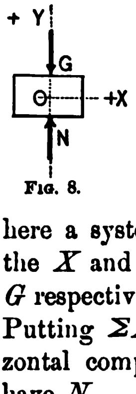


FIG. 8.

PROBLEM 2.—Fig. 9. A body of weight G (lbs.) is moving in a straight line over a rough horizontal table with a uniform velocity c (feet per second) to the right. The tension in an oblique cord by which it is pulled is given, and $= P$ (lbs.),

which remains constant, the cord making a given angle of elevation, α, with the path of the body. Required the vertical pressure N (lbs.) of the table, and also its horizontal action F (friction) (lbs.) against the body.

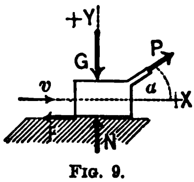


FIG. 9.

Referring by anticipation to Newton's first law of motion, viz., a material point acted on by no force or by balanced forces is either at rest or moving uniformly in a straight line, we see that this problem is a case of balanced forces, i.e., of equilibrium. Since there are only two unknown quantities, N and F, we may determine them by the two equations of Case I., taking the axes X and Y as before. Here let us leave the *direction* of N as well as its amount to be determined by the analysis. As F must evidently point toward the left, treat it as negative in summing the X components; the analysis, therefore, can be expected to give only its numerical value.

$\Sigma X = 0$ gives $P \cos \alpha - F = 0$. $\quad \therefore \; F = P \cos \alpha$.

$\Sigma Y = 0$ gives $N + P \sin \alpha - G = 0$. $\therefore \; N = G - P \sin \alpha$.

$\therefore$ N is upward or downward according as G is $>$ or $< P \sin \alpha$. For N to be a downward pressure upon the body would require the surface of the table to be above it. The ratio of the friction F to the pressure N which produces it can now be obtained, and is called the coefficient of friction. It may vary slightly with the velocity.

This problem may be looked upon as arising from an experiment made to determine the coefficient of friction between the given surfaces at the given uniform velocity.

19. The Free-Body Method.—The foregoing rather labored solutions of very simple problems have been made such to illustrate what may be called the free-body method of treating any problem involving a body acted on by a system of forces. It consists in conceiving the body isolated from all others which act on it in any way, those actions being [illegible]ed as so many forces, known or unknown, in amount [illegible]. The system of forces thus formed may be made [illegible] equa-

tions, whose character and number depend on circumstances, such as the behavior of the body, whether the forces are confined to a plane or not, etc., and which are therefore theoretically available for determining an equal number of unknown quantities, whether these be forces, masses, spaces, times, or abstract numbers. Of course in some instances the unknown quantities may enter these equations with such high powers that the elimination may be impossible; but this is a matter of algebra, not of mechanics.

CHAPTER II.

PARALLEL FORCES AND THE CENTRE OF GRAVITY.

20. Preliminary Remarks.—Although by its title this section should be restricted to a treatment of the equilibrium of forces, certain propositions involving the composition and resolution of forces, without reference to the behavior of the body under their action, will be found necessary as preliminary to the principal object in view.

As a rigid body possesses extension in three dimensions, to deal with a system of forces acting on it we require three co-ordinate axes: in other words, the system consists of "forces in space," and in general the forces are *non-concurrent.* In most problems in statics, however, the forces acting are in one plane: we accordingly begin by considering non-concurrent forces in a plane, of which the simplest case is that of two parallel forces. For the present the body on which the forces act will not be shown in the figure, but must be understood to be there (since we have no conception of forces independently of material bodies). The device will frequently be adopted of introducing into the given system two opposite and equal forces acting in the same line: evidently this will not alter the effect of the given system, as regards the rest or motion of the body.

21. Resultant of two Parallel Forces.

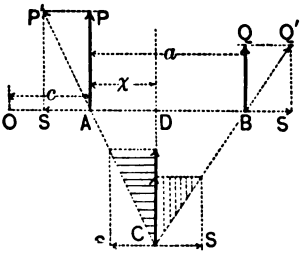


Fig. 10.

Case I.—The two forces have the *same direction.* Fig. 10. Let P and Q be the given forces, and AB a line perpendicular to them (P and Q are supposed to have been transferred to the intersection of A and B). Put in at A and ...d opposite forces S and S, combining them ... P'

and Q'. Transfer P' and Q' to their intersection at C, and there resolve them again into S and P, S and Q. S and S annul each other at C; therefore P and Q, acting along a common line CD, replace the P and Q first given; i.e., the resultant of the original two forces is a force $R = P + Q$, acting parallel to them through the point D, whose position must now be determined. The triangle CAD is similar to the triangle shaded by lines, $\therefore P : S :: \overline{CD} : x$; and CDB being similar to the triangle shaded by dots, $\therefore S : Q :: a - x : \overline{CD}$. Combining these, we have $\dfrac{P}{Q} = \dfrac{a-x}{x}$ and $\therefore\ x = \dfrac{Qa}{P+Q} = \dfrac{Qa}{R}$. Now write this $Rx = Qa$, and add Rc, i.e., $Pc + Qc$, to each member, c being the distance of O (Fig. 10), any point in AB produced, from A. This will give $R(x + c) = Pc + Q(a + c)$, in which c, $a + c$, and $x + c$ are respectively the lengths of perpendiculars let fall from O upon P, Q, and their resultant R. Any one of these products, such as Pc, is for convenience (since products of this form occur so frequently in Mechanics as a result of algebraic transformation) called the **Moment** of the force about the arbitrary point O. Hence the resultant of two parallel forces of the same direction is equal to their sum, acts in their plane, in a line parallel to them, and at such a distance from any arbitrary point O in their plane as may be determined by writing its moment about O equal to the sum of the moments of the two forces about O. O is called a *centre of moments*, and each of the perpendiculars a *lever-arm*.

Case II.—Two parallel forces P and Q of opposite directions. Fig. 11. By a process similar to the foregoing, we obtain $R = P - Q$ and $(P - Q)x = Qa$, i.e., $Rx = Qa$. Subtract each member of the last equation from Rc (i.e., $Pc - Qc$), in which c is the distance, from A, of any arbitrary point O in AB produced. This gives $R(c - x) = Pc - Q(a + c)$. But $(c - x)$, c, and $(a + c)$ are respectively the perpendiculars, from

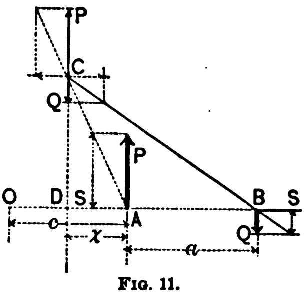


Fig. 11.

O, upon R, P, and Q. That is, $R(c - x)$ is the moment of R about O; Pc, that of P about O; and $Q(a + c)$, that of Q about O. But the moment of Q is subtracted from that of P, which corresponds with the fact that Q in this figure would produce a rotation about O opposite in direction to that of P. Having in view, then, this imaginary rotation, we may define the moment of a force as *positive* when the indicated direction about the given point is against the hands of a watch; as *negative* when with the hands of a watch.

Hence, in general, the resultant of any two parallel forces is, in amount, equal to their algebraic sum, acts in a parallel direction in the same plane, while its moment, about any arbitrary point in the plane, is equal to the algebraic sum of the moments of the two forces about the same point.

Corollary.—If each term in the preceding moment equations be multiplied by the secant of an angle (α, Fig. 12) thus:

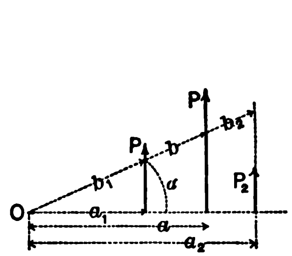


FIG. 12.

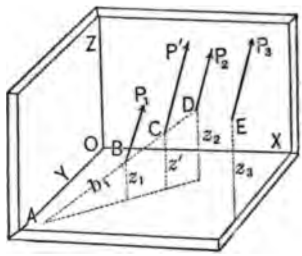


FIG. 13.

(using the notation of Fig. 12), we have $Pa \sec \alpha = P_1a_1 \sec \alpha + P_2a_2 \sec \alpha$, i.e., $Pb = P_1b_1 + P_2b_2$, in which b, b_1, and b_2 are the *oblique* distances of the three lines of action from any point O in their plane, and lie on the same straight line; P is the resultant of the parallel forces P_1 and P_2.

22. Resultant of any System of Parallel Forces in Space.—Let P_1, P_2, P_3, etc., be the forces of the system, and x_1, y_1, z_1, x_2, y_2, z_2, etc., the co-ordinates of ... ts of application as referred to an arbitrary set of t ... te axes X, Y, and Z, perpendicular to e ... is here re-

stricted to a definite point of application in its line of action (with reference to establishing more directly the fundamental equations for the co-ordinates of the centre of gravity of a body). The resultant P' of any two of the forces, as P_1 and P_2, is $= P_1 + P_2$, and may be applied at C, the intersection of its own line of action with a line BD joining the points of application of P_1 and P_2, its components. Produce the latter line to A, where it pierces the plane XY, and let b_1, b', and b_2, respectively, be the distances of B, C, D, from A; then from the corollary of the last article we have

$$P'b' = P_1b_1 + P_2b_2;$$

but from similar triangles

$$b' : b_1 : b_2 :: z' : z_1 : z_2, \qquad \therefore\ P'z' = P_1z_1 + P_2z_2.$$

Now combine P', applied at C, with P_3, applied at E, calling their resultant P'' and its vertical co-ordinate z'', and we obtain

$$P''z'' = P'z' + P_3z_3, \text{ i.e., } P''z'' = P_1z_1 + P_2z_2P_3z_3,$$

also

$$P'' = P' + P_3 = P_1 + P_2 + P_3.$$

Proceeding thus until all the forces have been considered, we shall have finally, for the resultant of the whole system,

$$R = P_1 + P_2 + P_3 + \text{etc.};$$

and for the vertical co-ordinate of its point of application, which we may write $\bar{z}$,

$$R\bar{z} = P_1z_1 + P_2z_2 + P_3z_3 + \text{etc};$$

$$\text{i.e., } \bar{z} = \frac{P_1z_1 + P_2z_2 + P_3z_3 \ldots}{P_1 + P_2 + P_3 + \ldots.} = \frac{\Sigma(Pz)}{\Sigma P};$$

and similarly for the other co-ordinates.

$$\bar{x} = \frac{\Sigma(Px)}{\Sigma P} \text{ and } \bar{y} = \frac{\Sigma(Py)}{\Sigma P}.$$

In these equations, in the general case, such products as P_1z_1, etc., cannot strictly be called moments. The point whose co-

ordinates are the $\bar{x}$, $\bar{y}$, and $\bar{z}$, just obtained, is called the *Centre of Parallel Forces*, and its position is *independent of the* (*common*) *direction* of the forces concerned.

Example.—If the parallel forces are contained in one plane, and the axis Y be assumed parallel to the direction of the forces, then each product like P_1x_1 will be a *moment*, as defined in § 21; and it will be noticed in the accompanying numerical example, Fig. 14, that a detailed substitution in the equation

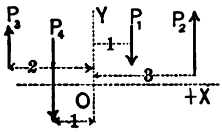


FIG. 14.

$$R\bar{x} = P_1x_1 + P_2x_2 + \text{etc.}, \quad . \quad . \quad . \quad (1)$$

having regard to the proper sign of each force and of each abscissa, gives the same result as if each product Px were first obtained numerically, and a sign affixed to the product considered as a moment about the point O. Let $P_1 = -1$ lb.; $P_2 = +2$ lbs.; $P_3 = +3$ lbs.; $P_4 = -6$ lbs.; $x_1 = +1$ ft.; $x_2 = +3$ ft.; $x_3 = -2$ ft.; and $x_4 = -1$ ft. Required the amount and position of the resultant R. In amount $R = \Sigma P = -1 + 2 + 3 - 6 = -2$ lbs.; i.e., it is a *downward* force of 2 lbs. As to its position, $R\bar{x} = \Sigma(Px)$ gives $(-2)\bar{x} = (-1) \times (+1) + 2 \times 3 + 3 \times (-2) + (-6) \times (-1) = -1 + 6 - 6 + 6$. Now from the figure, by inspection, it is evident that the moment of P_1 about O is negative (*with* the hands of a watch), and is numerically $= 1$, i.e., its moment $= -1$; similarly, by inspection, that of P_2 is seen to be positive, that of P_3 negative, that of P_4 positive; which agree with the results just found, that $(-2)\bar{x} = -1 + 6 - 6 + 6 = +5$ ft. lbs. (Since a moment is a product of a force (lbs.) by a length (ft.), it may be called so many foot-pounds.) Next, solving for $\bar{x}$, we obtain $\bar{x} = (+5) \div (-2) = -2.5$ ft.; i.e., the resultant of the given forces is a downward force of 2 lbs. acting in a vertical line 2.5 ft. to the left of the origin. Hence, if the body in question be a horizontal rod whose weight has been already included in the statement of forces, a support placed 2.5 ft. to the left of O and capable of resisting at least 2 lbs. downward pressure will preserve equilibrium; and the pressure which it exerts

against the rod must be an upward force, P_5, of 2 lbs., i.e., the equal and opposite of the resultant of P_1, P_2, P_3, P_4.

Fig. 15 shows the rod as a free body in equilibrium under the five forces. $P_5 = +2$ lbs. $=$ the *reaction* of the support. Of course $P_5 =$ is one of a pair of equal and opposite forces; the other one is the pressure of the rod against the support, and would take its place among the forces acting on the support.

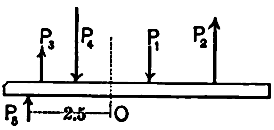


FIG. 15.

23. Centre of Gravity.—Among the forces acting on any rigid body at the surface of the earth is the so-called attraction of the latter (i.e., gravitation), as shown by a spring-balance, which indicates the *weight* of the body hung upon it. The weights of the different particles of any rigid body constitute a system of parallel forces (practically so, though actually slightly convergent). The point of application of the resultant of these forces is called the *centre of gravity* of the body, and may also be considered the *centre of mass*, the body being of very small dimensions compared with the earth's radius.

If $\bar{x}$, $\bar{y}$, and $\bar{z}$ denote the co-ordinates of the centre of gravity of a body referred to three co-ordinate axes, the equations derived for them in § 22 are directly applicable, with slight changes in notation.

Denote the weight of any particle of the body by dG, its volume by dV, and its *heaviness* (rate of weight, see § 7) and its co-ordinates by x, y, and z; then, using the integral sign as indicating a summation of like terms for all the particles of the body, we have, for heterogeneous bodies,

$$\bar{x} = \frac{\int \gamma x dV}{\int \gamma dV}; \quad \bar{y} = \frac{\int \gamma y dV}{\int \gamma dV}; \quad \bar{z} = \frac{\int \gamma z dV}{\int \gamma dV}; \quad . \quad . \quad (1)$$

while, if the body is homogeneous, γ is the same for all its elements, and being therefore placed outside the sign of summation, is cancelled out, leaving for *homogeneous* bodies (V denoting the total volume)

$$\bar{x} = \frac{\int x dV}{V}; \quad \bar{y} = \frac{\int y dV}{V}; \quad \text{and } \bar{z} = \frac{\int z dV}{V}. \quad . \quad . \quad (2)$$

Corollary.—It is also evident that if a homogeneous body is for convenience considered as made up of several finite parts, whose volumes are V_1, V_2, etc., and whose gravity co-ordinates are $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$; $\bar{x}_2$, $\bar{y}_2$, $\bar{z}_2$; etc., we may write

$$\bar{x} = \frac{V_1\bar{x}_1 + V_2\bar{x}_2 + \ldots}{V_1 + V_2 + \ldots} \quad . \quad . \quad . \quad . \quad (3)$$

If the body is heterogeneous, put G_1 (weights), etc., instead of V_1, etc., in equation (3).

If the body is an infinitely thin *homogeneous shell* of uniform thickness $= h$, then $dV = hdF$ (dF denoting an element, and F the whole area of one surface) and equations (2) become, after cancellation,

$$\bar{x} = \frac{\int xdF}{F}; \quad \bar{y} = \frac{\int ydF}{F}; \quad \bar{z} = \frac{\int zdF}{F}. \quad . \quad . \quad . \quad (4)$$

Similarly, for a *homogeneous wire* of infinitely small cross-section (i.e., a geometrical line, having weight), its length being s, and an element of length ds, we obtain

$$\bar{x} = \frac{\int xds}{s}; \quad \bar{y} = \frac{\int yds}{s}; \quad \bar{z} = \frac{\int zds}{s}. \quad . \quad . \quad . \quad (5)$$

It is often convenient to find the centre of gravity of a thin plate by experiment, balancing it on a needle-point; other shapes are not so easily dealt with.

24. Symmetry.—Considerations of symmetry of form often determine the centre of gravity of homogeneous solids without analysis, or limit it to a certain line or plane. Thus the centre of gravity of a sphere, or any regular polyedron, is at its centre of figure; of a right cylinder, in the middle of its axis; of a thin pl[illegible] of the form o[illegible]ircle or regular polygon, in the centre o[illegible]re; of a stra[illegible]ire of uniform cross-section, in [illegible] midd[illegible]s length.

[illegible] symmetrical about a plane, [illegible]t plane, called a plane of

gravity; if about a line, in that line called a line of gravity; if about a point, in that point.

25. By considering certain modes of subdivision of a homogeneous body, lines or planes of gravity are often made apparent. E.g., a line joining the middle of the bases of a trapezoidal plate is a line of gravity, since it bisects all the strips of uniform width determined by drawing parallels to the bases; similarly, a line joining the apex of a triangular plate to the middle of the opposite side is a line of gravity. Other cases can easily be suggested by the student.

26. Problems.—(1) Required the position of the centre of gravity of a *fine homogeneous wire of the form of a circular arc, AB*, Fig. 16. Take the origin O at the centre of the circle, and the axis X bisecting the wire. Let the length of the wire, s, $= 2s_1$; $ds =$ element of arc. We need determine only the $\bar{x}$, since evidently $\bar{y} = 0$. Equations (5), § 23, are applicable here, i.e., $\bar{x} = \frac{\int x ds}{s}$.

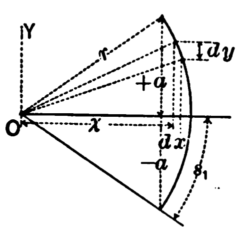


FIG. 16.

From similar triangles we have

$$ds : dy :: r : x; \therefore ds = \frac{rdy}{x};$$

$$\therefore \bar{x} = \frac{r}{2s_1}\int_{y=-a}^{y=+a} dy = \frac{2ra}{2s_1},$$ i.e., = chord × radius ÷ length of wire. For a semicircular wire, this reduces to $\bar{x} = 2r \div \pi$.

PROBLEM 2. *Centre of gravity of trapezoidal (and triangular) thin plates, homogeneous, etc.*—Prolong the non-parallel sides of the trapezoid to intersect at O, which take as an origin, making the axis X perpendicular to the bases b and b_1. We may here use equations (4), § 23, and may take a vertical strip for our element of area, dF, in determining $\bar{x}$; for each point of such a strip has the same x. Now $dF = (y + y')dx$, and

from similar triangles $y + y' = \frac{b}{h}x$. Hence $F = \frac{1}{2}(bh - b_1h_1)$ can be written $\frac{1}{2}\frac{b}{h}(h^2 - h_1^2)$, and $\bar{x} = \frac{\int x dF}{F}$ becomes

$$= \left[\frac{b}{h}\int_{h_1}^{h} x^2 dx\right] \div \frac{1}{2}\frac{b}{h}(h^2 - h_1^2) = \frac{2}{3}\frac{h^3 - h_1^3}{h^2 - h_1^2}$$

for the trapezoid.

For a triangle $h_1 = 0$, and we have $\bar{x} = \frac{2}{3}h$; that is, the centre of gravity of a triangle is one third the altitude from the base. The centre of gravity is finally determined by knowing

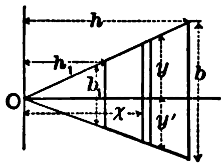


FIG. 17.

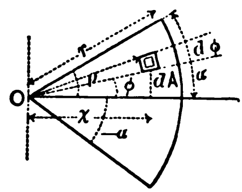


FIG. 18.

that a line joining the middles of b and b_1 is a line of gravity; or joining O and the middle of b in the case of a triangle.

PROBLEM 3. *Sector of a circle. Thin plate,* etc.—Let the notation, axes, etc., be as in Fig. 18. Angle of sector $= 2\alpha$; $\bar{x} = ?$ Using polar co-ordinates, the element of area dF (a small rectangle) $= \rho d\varphi \,.\, d\rho$, and its $x = \rho \cos \varphi$; hence the total area $=$

$$F = \int_{-\alpha}^{+\alpha}\left[\int_0^r \rho d\rho\right] d\varphi = \int_{-\alpha}^{+\alpha}\frac{1}{2}r^2 d\varphi = \frac{r^2}{2}\left[\varphi\right]_{-\alpha}^{+\alpha};$$

i.e., $F = r^2\alpha$. From equations (5), § 23, we have

$$\bar{x} = \frac{1}{F}\int x dF$$

$$= \frac{1}{F}\int\int \cos \varphi\rho^2 d\rho d\varphi = \frac{1}{F}\int_{-\alpha}^{+\alpha}\left[\cos\varphi \int_0^r \rho^2 d\rho\right] d\varphi.$$

(*Note on double integration.*—The quantity

$$\left[\cos\varphi \int_0^r \rho^2\, d\rho\right] d\varphi,$$

is that portion of the summation $\int\int \cos\varphi\rho^2 d\rho d\varphi$ which belongs to a single elementary sector (triangle), since all its elements (rectangles), from centre to circumference, have the same φ and $d\varphi$.)

That is,

$$\bar{x} = \frac{1}{F}\cdot\frac{r^3}{3}\int_{-\alpha}^{+\alpha}\cos\varphi d\varphi = \frac{r^3}{3r^2\alpha}\left[\sin\varphi\right]_{-\alpha}^{+\alpha} = \frac{2}{3}\cdot\frac{r\sin\alpha}{\alpha};$$

or, putting $\beta = 2\alpha =$ total angle of sector, $\bar{x} = \frac{4}{3}\,\frac{r\sin\frac{1}{2}\beta}{\beta}$.

For a semicircular plate this reduces to $\bar{x} = \frac{4r}{3\pi}$.

[*Note.*—In numerical substitution the arcs α and β used above (unless sin or cos is prefixed) are understood to be expressed in circular measure (π-measure); e.g., for a quadrant, $\beta = \frac{\pi}{2} = 1.5707+$; for 30°, $\beta = \frac{\pi}{6}$; or, in general, if β in degrees $= \frac{180°}{n}$, then β in π-measure $= \frac{\pi}{n}$.]

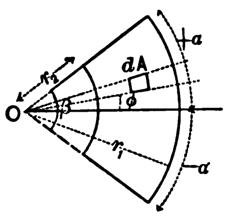


FIG. 19.

PROBLEM 4. *Sector of a flat ring*; thin plate, etc.—Treatment similar to that of Problem 3, the difference being that the limits of the interior integrations are $\left[_{r_2}^{r_1}\right.$ instead of $\left[_{0}^{r}\right.$. Result,

$$\bar{x} = \frac{4}{3}\cdot\frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}\cdot\frac{\sin\frac{1}{2}\beta}{\beta}.$$

PROBLEM 5.—*Segment of a circle;* thin plate, etc.—Fig. 20. Since each rectangular element of any vertical strip has the same x, we may take the strip as dF in finding $\bar{x}$, and use y as the half-height of the strip. $dF = 2ydx$, and from similar triangles $x : y :: (-dy) : dx$, i.e., $xdx = -ydy$. Hence from eq. (5), § 23,

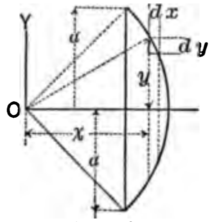


FIG. 20.

$$\bar{x} = \frac{\int x dF}{F} = \frac{\int x 2y dx}{F} = \frac{-2\int_a^0 y^2 dy}{F} = \frac{2}{3F}\Big[(-y^3)\Big]_a^0 = \frac{2}{3}\cdot\frac{a^3}{F};$$

but $a =$ the half-chord, hence, finally, $\bar{x} = \dfrac{(chord)^3}{12F}$.

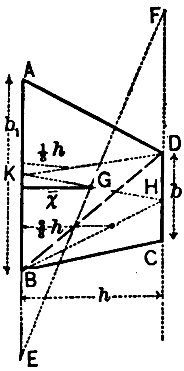


FIG. 21.

PROBLEM 6.—*Trapezoid;* thin plate, etc., by the method in the corollary of § 23; equations (3). Required the distance $\bar{x}$ from the base AB. Join DB, thus dividing the trapezoid $ABCD$ into two triangles $ADB = F_1$ and $DBC = F_2$, whose gravity x's are, respectively, $x_1 = \frac{1}{3}h$ and $x_2 = \frac{2}{3}h$. Also, $F_1 = \frac{1}{2}hb_1$, $F_2 = \frac{1}{2}hb_2$, and F (area of trapezoid) $= \frac{1}{2}h(b_1 + b_2)$. Eq. (3) of § 23 gives $F\bar{x} = F_1x_1 + F_2x_2$; hence, substituting $(b_1 + b_2)\,\bar{x} = \frac{1}{3}b_1h + \frac{2}{3}b_2h$.

$$\therefore \bar{x} = \frac{h}{3}\cdot\frac{(b_1 + 2b_2)}{b_1 + b_2}.$$

The line joining the middles of b_1 and b_2 is a line of gravity, and is divided in such a ratio by the centre of gravity that the following construction for finding the latter holds good: Prolong each base, in opposite directions, an amount equal to the other base; join the two points thus found: the intersection with the other line of gravity is the centre of gravity of the trapezoid. Thus, Fig. 21, with $BE = b_2$ and $DF = b_1$, join FE, etc.

PROBLEM 7. *Homogeneous oblique cone or pyramid.*— Take the origin at the vertex, and the axis X perpendicular to the base (or bases, if a frustum). In finding $\bar{X}$ we may put $dV =$ volume of any lamina parallel to YZ, F being the base of such a lamina, each point of the lamina having the same x. Hence, (equations (2), § 23),

$$\bar{x} = \frac{1}{V}\int x dV, \quad V = \int dV = \int F dx;$$

but

$$F : F_2 :: x^2 : h_2^2, \quad \therefore F = \frac{F_2}{h_2^2}x^2,$$

and

$$V = \frac{F_2}{h_2^2}\int x^2 dx = \frac{F_2}{h_2^2}\left[\frac{x^3}{3}\right. \cdot \int x dV = \frac{F_2}{h_2^2}\int x^3 dx = \frac{F_2}{h_2^2}\left[\frac{x^4}{4}\right. .$$

For a frustum, $x = \frac{3}{4}\cdot\frac{h_2^4 - h_1^4}{h_2^3 - h_1^3}$; while for a pyramid, h_1 being $= 0$, $\bar{x} = \frac{3}{4}h$. Hence the centre of gravity of a pyramid is one fourth the altitude from the base. It also lies in the line joining the vertex to the centre of gravity of the base.

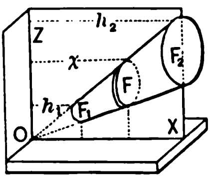


FIG. 22.

PROBLEM 8.—If the heaviness of the material of the above cone or pyramid varied directly as x, γ_2 being its heaviness at the base F_2, we would use equations (1), § 23, putting $\gamma = \frac{\gamma_2}{h_2}x$; and finally, for the frustum,

$$\bar{x} = \frac{4}{5}\cdot\frac{h_2^5 - h_1^5}{h_2^4 - h_1^4},$$

and for a complete cone $\bar{x} = \frac{4}{5}h_2$.

27. The Centrobaric Method.—If an elementary area dF be revolved about an axis in its plane, through an angle $\alpha < 2\pi$,

the distance from the axis being $= x$, the volume generated is $dV = \alpha x dF$, and the total volume generated by all the dF's of a finite plane figure whose plane contains the axis and which lies entirely on one side of the axis, will be $V = \int dV = \alpha \int x dF$. But from § 23, $\alpha \int x dF = \alpha F \bar{x}$; $\alpha \bar{x}$ being the length of path described by the centre of gravity of the plane figure, we may write: *The volume of a solid of revolution generated by a plane figure, lying on one side of the axis, equals the area of the figure multiplied by the length of curve described by the centre of gravity of the figure.*

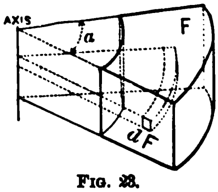


FIG. 23.

A corresponding statement may be made for the surface generated by the revolution of a line. The arc α must be expressed in π measure in numerical work.

27a. Centre of Gravity of any Quadrilateral.—Fig. 23*a*. *Construction;* $ABGD$ being any quadrilateral. Draw the diagonals. On the long segment DK of DB lay off $DE = BK$, the shorter, to determine E; similarly, determine N on the other diagonal, by making $GN = AK$. Bisect EK in H and KN in M. The intersection of EM and NH is the centre of gravity, C.

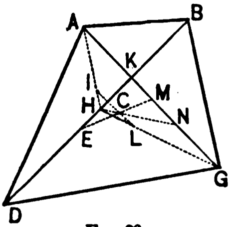


FIG. 23*a*.

Proof.—H being the middle of DB, and AH and HG having been joined, I the centre of gravity of the triangle ABD is found on AH, by making $HI = \frac{1}{3}AH$; similarly, by making $HL = \frac{1}{3}HG$, L is the centre of gravity of triangle BDG. $\therefore$ IL is parallel to AG and is a gravity-line of the whole figure; and the centre of gravity C may be found on it if we can make $CL : CI ::$ area ABD : area BDG (§ 21). But since these triangles have a common base DB, their areas are proportional to the slant heights (equally inclined to DB) AK and KG, i.e., to GN and NA. Hence HN, which divides IL in the required ratio, contains C, and *is* $\therefore$ *a gravity-line.* By similar reasoning, using the other diagonal, AG, and

the two triangles into which it divides the whole figure, we may prove EM to be a gravity-line also. Hence the construction is proved.

27b. EXAMPLES.—1. Required the volume of a sphere by the centrobaric method.

A sphere may be generated by a semicircle revolving about its diameter through an arc $\alpha = 2\pi$. The length of the path described by its centre of gravity is $= 2\pi \frac{4r}{3\pi}$ (see Prob. 3, § 26), while the area of the semicircle is $\frac{1}{2}\pi r^2$. Hence by § 27,

$$\text{Volume generated} = 2\pi \,.\, \frac{4r}{3\pi} \,.\, \tfrac{1}{2}\pi r^2 = \frac{4}{3}\pi r^3.$$

2. Required the position of the centre of gravity of the sector of a flat ring in which $r_1 = 21$ feet, $r_2 = 20$ feet, and $\beta = 80°$ (see Fig. 19, and § 26, Prob. 4).

$\sin \frac{\beta}{2} = \sin 40° = 0.64279$, and β in *circular measure* $= \frac{80}{180}\pi = \frac{4}{9}\pi = 1.3962$. By using r_1 and r_2 in feet, $\bar{x}$ will be obtained in feet.

$$\therefore \bar{x} = \frac{4}{3} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \cdot \frac{\sin \frac{\beta}{2}}{\beta} = \frac{4}{3} \cdot \frac{1261}{41} \cdot \frac{0.64279}{1.3962} = 18.87 \text{ feet.}$$

CHAPTER III.

STATICS OF A RIGID BODY.

28. Couples.—On account of the peculiar properties and utility of a system of two equal forces acting in parallel lines and in opposite directions, it is specially considered, and called a **Couple.** The *arm* of a couple is the perpendicular distance between the forces; its *moment*, the product of this arm, by one of the forces. The axis of a couple is an imaginary line drawn perpendicular to its plane on that side from which the rotation appears *positive* (against the hands of a watch). (An ideal rotation is meant, suggested by the position of the arrows; any actual rotation of the rigid body is a subject for future consideration.) In dealing with two or more couples the lengths of their axes are made proportional to their moments; in fact, by selecting a proper scale, numerically equal to these moments. E.g., in Fig. 24, the moments of the two couples there shown are Pa and Qb; their axes p and q so laid off that $Pa : Qb :: p : q$, and that the ideal rotation may appear positive, viewed from the outer end of the axis.

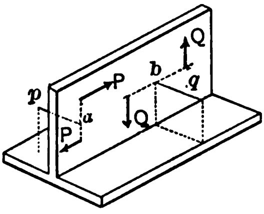


FIG. 24.

29. *No single force can balance a couple.*—For suppose the couple P, P, could be balanced by a force R', then this, acting at some point C, ought to hold the couple in equilibrium. Draw CO through O, the **centre** of **symmetry** of the couple, and **make** [illegible] At D put in two opposite and **equal** [illegible] and parallel to R'. The supposed [illegible] R', P, and

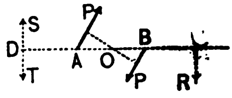


FIG. 25.

P are in equilibrium, so ought (by symmetry about O) S, P, and P to be in equilibrium, and they may be removed without disturbing equilibrium. But we have left T and R', which are evidently not in equilibrium; $\therefore$ the proposition is proved by this *reductio ad absurdum.* Conversely a couple has no single resultant.

30. *A couple may be transferred anywhere in its own plane.*—First, it may be turned through any angle α, about any point of its arm, or of its arm produced.

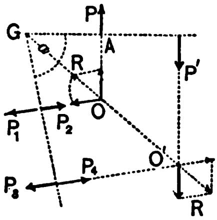


FIG. 26.

Let (P, P') be a couple, G any point of its arm (produced), and α any angle. Make $GC = GA$, $CD = AB$, and put in at C, P_1 and P_2 equal to P (or P'), opposite to each other and perpendicular to GC; and P_3 and P_4 similarly at D. Now apply and combine P and P_1 at O, P' and P_4 at O'; then evidently R and R' neutralize each other, leaving P_2 and P_3 equivalent to the original couple (P, P'). The arm $\overline{CD} = AB$. Secondly, if G be at infinity, and $\alpha = 0$, the same proof applies, i.e., a couple may be moved parallel to itself in its own plane. Therefore, by a combination of the two transferrals, the proposition is established for any transferral in the plane.

31. *A couple may be replaced by another of equal moment in a parallel plane.*—Let (P, P') be a couple. Let CD, in a parallel plane, be parallel to AB. At D put in a pair of equal and opposite forces, S_3 and S_4, parallel to P and each $= \dfrac{\overline{AE}}{\overline{ED}}P.$

Similarly at C, S_1 and S_2 parallel to P and each $= \dfrac{\overline{BE}}{\overline{EC}}P.$

But, from similar triangles,

$$\frac{AE}{ED} = \frac{BE}{EC}; \therefore S_1 = S_2 = S_3 = S_4.$$

[NOTE.—The above values are so chosen that the intersection point E may be the point of application of $(P' + S_2)$, the resultant of P' and S_2; and also of $(P + S_3)$, the resultant of P and S_3, as follows from § 21; thus (Fig. 28), R, the resultant of the two parallel forces P and S_3, is $= P + S_3$, and its moment about any centre of moments, as E, its own point of application, should equal the (algebraic) sum of the moments of its components about E; i.e., $R \times \text{zero} = P . \overline{AE} - S_3 . \overline{DE}$; $\therefore S_3 = \dfrac{\overline{AE}}{\overline{DE}} . P$.]

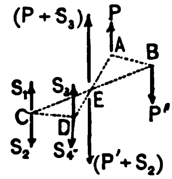


FIG. 27.

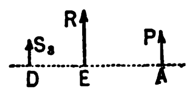


FIG. 28.

Replacing P' and S_2 by $(P' + S_2)$, and P and S_3 by $(P + S_3)$, the latter resultants cancel each other at E, leaving the couple (S_1, S_4) with an arm CD, equivalent to the original couple P, P' with an arm AB. But, since $S_1 = \dfrac{\overline{BE}}{\overline{EC}} . P = \dfrac{\overline{AB}}{\overline{CD}} . P$, we have $S_1 \times \overline{CD} = P \times \overline{AB}$; that is, their moments are equal.

32. Transferral and Transformation of Couples.—In view of the foregoing, we may state, in general, that a couple acting on a rigid body may be transferred to any position in any parallel plane, and may have the values of its forces and arm changed in any way so long as its moment is kept unchanged, and still have the same effect on the rigid body (as to rest or motion, not in distorting it).

Corollaries.—A couple may be replaced by another in any position so long as their axes are equal and parallel and similarly situated with respect to their planes.

A couple can be balanced only by another couple whose axis is equal and parallel to that of the first, and dissimilarly situated. For example, Fig. 29, Pa being $= Qb$, the rigid body AB (here supposed without weight) is in equilibrium in each

case shown. By "reduction of a couple to a certain arm a" is meant that for the original couple whose arm is a', with forces each $= P'$, a new couple is substituted whose arm shall be $= a$, and the value of whose forces P and P must be computed from the condition

$$Pa = P'a', \quad \text{i.e.,} \quad P = P'a' \div a.$$

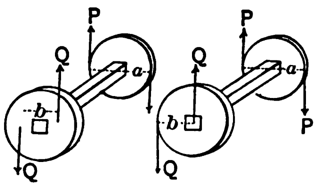


FIG. 29.

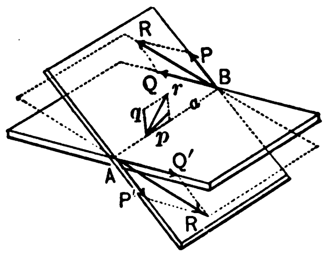


FIG. 30.

33. Composition of Couples.—Let (P, P') and (Q, Q') be two couples in different planes *reduced to the same arm* $\overline{AB} = a$, which is a portion of the line of intersection of their planes. That is, whatever the original values of the individual forces and arms of the two couples were, they have been transferred and replaced in accordance with § 32, so that $P.\overline{AB}$, the moment of the first couple, and the direction of its axis, p, have remained unchanged; similarly for the other couple. Combining P with Q and P' with Q', we have a resultant couple (R, R') whose arm is also $\overline{AB}$. The axes p and q of the component couples are proportional to $P.\overline{AB}$ and $Q.AB$, i.e., to P and Q, and contain the same angle as P and Q. Therefore the parallelogram $p \ldots q$ is similar to the parallelogram $P \ldots Q$; whence $p : q : r :: P : Q : R$, or $p : q : r :: Pa : Qa : Ra$. Also r is evidently perpendicular to the plane of the resultant couple (R, R'), whose moment is Ra. Hence r, the diagonal of the parallelogram on p and q, is the axis of the resultant couple. To combine two couples, therefore, we have only to combine their axes, as if they were forces, by a parallelogram, the diagonal being the axis of the resultant couple; the plane of this couple will be perpendicular to the

axis just found, and its moment bears the same relation to the moments of the component couples as the diagonal axis to the two component axes. Thus, if two couples, of moments Pa and Qb, lie in planes perpendicular to each other, their resultant couple has a moment $Rc = \sqrt{(Pa)^2 + (Qb)^2}$.

If three couples in different planes are to be combined, the axis of their resultant couple is the diagonal of the parallelopiped formed on the axes, laid off to the same scale and *pointing in the proper directions*, the proper *direction* of an axis being *away* from the plane of its couple, on the side from which the couple appears of positive direction.

34. If several couples lie in the same plane their axes are parallel and the axis of the resultant couple is their algebraic sum; and a similar relation holds for the moments: thus, in Fig. 24, the resultant of the two couples has a moment $= Qb - Pa$, which shows us that a convenient way of combining couples, when all in one plane, is to call the moments positive or negative, according as the ideal rotations are against, or with, the hands of a watch, as seen from the *same* side of the plane; the sign of the algebraic sum will then show the ideal rotation of the resultant couple.

35. Composition of Non-concurrent Forces in a Plane.—Let P_1, P_2, etc., be the forces of the system; x_1, y_1, x_2, y_2, etc., the

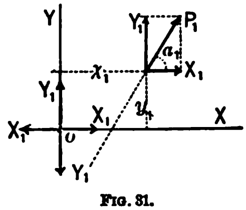


FIG. 31.

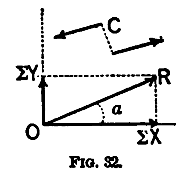


FIG. 32.

co-ordinates of their points of application; and α_1, α_2, . . . etc., their angles with the axis X. Replace P_1 by its components X_1 and Y_1, parallel to the arbitrary axes of reference. At the origin put in two forces, opposite to each other and equal and parallel to X_1; similarly for Y_1. (Of course $X_1 = P_1 \cos \alpha$ and $Y_1 = P_1 \sin \alpha$.) We now have P_1 replaced by two forces X_1

and Y_1 *at the origin*, and two couples, in the same plane, whose moments are respectively $-X_1y_1$ and $+Y_1x_1$, and are therefore (§ 34) equivalent to a single couple, in the same plane with a moment $= (Y_1x_1 - X_1y_1)$.

Treating all the remaining forces in the same way, the whole system of forces is replaced by

the force $\Sigma(X) = X_1 + X_2 + \dots$ at the origin, along the axis X;
the force $\Sigma(Y) = Y_1 + Y_2 + \dots$ at the origin, along the axis Y;

and the couple whose moment $= \Sigma(Yx - Xy)$, which may be called the couple C (see Fig. 32), and may be placed anywhere in the plane. Now $\Sigma(X)$ and (Y) may be combined into a force R; i.e.,

$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2}$ and its direction-cosine is $\cos \alpha = \dfrac{\Sigma X}{R}$.

Since, then, the whole system reduces to C and R, we must have for equilibrium $R = 0$, and $G = 0$; i.e., **for equilibrium**

$$\Sigma X = 0,\ \Sigma Y = 0, \text{ and } \Sigma(Yx - Xy) = 0. \quad . \text{ eq. (1)}$$

If R alone $= 0$, the system reduces to a couple whose moment is $G = \Sigma(Yx - Xy)$; and if G alone $= 0$ the system reduces to a single force R, applied at the origin. If, in general, neither R nor $G = 0$, the system is still equivalent to a single force, but not applied at the origin (as could hardly be expected, since the origin is arbitrary); as follows (see Fig. 33):

Replace the couple C by one of equal moment, G, with each force $= R$. Its arm will therefore be $\dfrac{G}{R}$. Move this couple in the plane so that one of its forces R may cancel the R already at the origin, thus leaving a **single resultant** R for the whole system, applied in a line at a perpendicular distance, $c = \dfrac{G}{R}$, from the origin, and making an angle α whose cosine $= \dfrac{\Sigma X}{R}$, with the axis X.

36. More convenient form for the equations of equilibrium of non-concurrent forces in a plane.—In (I.), Fig. 34, O being

any point and a its perpendicular distance from a force P; put in at O two equal and opposite forces P and P' each equal to P, and we have P replaced by an equal single force P' at O, and a couple whose moment is $+ Pa$. (II.) shows a similar construction, dealing with the X and Y components of P, so that in (II.) P is replaced by single forces X' and Y' at O

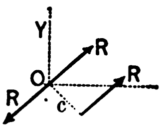


FIG. 33.

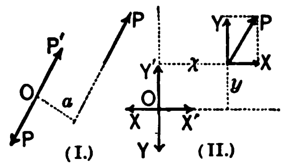


FIG. 34.

(and they are equivalent to a resultant P', at O, as in (I.), and two couples whose moments are $+ Yx$ and $- Xy$.

Hence, O being the same point in both cases, the couple Pa is equivalent to the two last mentioned, and, their axes being parallel, we must have $Pa = Yx - Xy$. Equations (1), § 35, for equilibrium, may now be written

$$\Sigma X = 0, \ \Sigma Y = 0, \text{ and } \Sigma(Pa) = 0. \quad . \quad . \quad (2)$$

In problems involving the equilibrium of non-concurrent forces in a plane, we have *three independent conditions*, or *equations*, and can determine at most three unknown quantities. For practical solution, then, the rigid body having been made *free* (by conceiving the actions of all other bodies as represented by forces), and being in equilibrium (which it must be if at rest), we apply equations (2) literally; i.e., assuming an origin and two axes, equate the sum of the X components of all the forces to zero; similarly for the Y components; and then for the "moment-equation," having dropped a perpendicular from the origin upon each force, write the algebraic sum of the products (*moments*) obtained by multiplying each force by its perpendicular, or "*lever-arm*," equal to zero, calling each product + or − acc[illegible]g as the ideal rotation appears against, or with, the han[illegible] watch, as seen from the same side of the plane. (...e convention would do as

Sometimes it is convenient to use three moment equations, taking a new origin each time, and then the $\Sigma X = 0$ and $\Sigma Y = 0$ are superfluous, as they would not be independent equations.

37. Problems involving Non-concurrent Forces in a Plane.—*Remarks.* The weight of a rigid body is a vertical force through its centre of gravity, downwards.

If the surface of contact of two bodies is *smooth* the action (pressure, or force) of one on the other is perpendicular to the surface at the point of contact. If a cord must be imagined cut, to make a body free, its tension must be inserted in the line of the cord, and in such a direction as to keep *taut* the small portion still fastened to the body. In case the pin of a hinge must be removed, to make the body free, its pressure against the ring being unknown in *direction* and *amount*, it is most convenient to represent it by its unknown components X and Y, in *known* directions. In the following problems there is supposed to be no friction. If the line of action of an unknown force is known, but not its direction (forward or backward), assume a direction for it and adhere to it in all the three equations, and if the assumption is correct the value of the force, after elimination, will be positive; if incorrect, negative.

Problem 1.—Fig. 35. Given an oblique rigid rod, with two loads G_1 (its own weight) and G_2; required the reaction of the *smooth* vertical wall at A, and the direction and amount of the *hinge*-pressure at O. The reaction at A must be horizontal; call it X'. The pressure at O, being unknown in direction, will have both its X and Y components unknown. The three unknowns, then, are X_0, X', and Y_0, while G_1, G_2, a_1, a_2, and h are known. The figure shows the rod as a *free body*, all the forces acting on it have been put in, and, since the rod is at rest, constitute a system of non-concurrent forces in a plane, ready for the conditions of equilibrium. Taking origin and axes as in the figure,

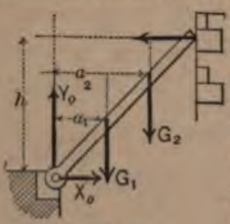


FIG. 35.

$\Sigma X = 0$ gives $+ X_0 - X' = 0$; $\Sigma Y = 0$ gives $+ Y_0 - G_1 - G_2 = 0$; while $\Sigma(Pa) = 0$, about O, gives $+ X'h - G_1a_1 - G_2a_2 = 0$. (The moments of X_0 and Y_0 about O are, each, $=$ zero.) By elimination we obtain $Y_0 = G_1 + G_2$; $X_0 = X' = [G_1a_1 + G_2a_2] \div h$; while the pressure at $O = \sqrt{X_0^2 + Y_0^2}$, and makes with the horizontal an angle whose tan $= Y_0 \div X_0$.

[N.B. A special solution for this problem consists in this, that the resultant of the two known forces G_1 and G_2 intersects the line of X' in a point which is easily found by § 21. The hinge-pressure must pass through this point, since three forces in equilibrium must be concurrent.]

We might vary this problem by limiting X' to a safe value, depending on the stability of the wall, and making h an unknown. The three unknowns would then be X_0, Y_0, and h.

Problem 2.—Given two rods with loads, three hinges (or "pin-joints"), and all dimensions: required the three hinge-

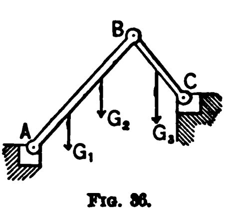


FIG. 36.

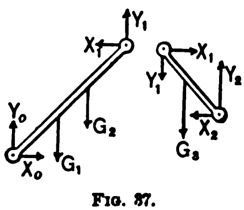


FIG. 37.

pressures; i.e., there are six unknowns, viz., three X and three Y components. We obtain three equations from each of the two free bodies in Fig. 37. The student may fill out the details. Notice the application of the principles of action and reaction at B (see § 3).

Problem 3.—A Warren bridge-truss rests on the horizontal *smooth* abutment-surfaces in Fig. 38. It is composed of equal isosceles triangles; no piece is continuous beyond a joint, each **of which is a *pin connection*.** All **loads are** [illegible] **as acting at the joints,** [illegible] **ece will be** [illegible] **or c**

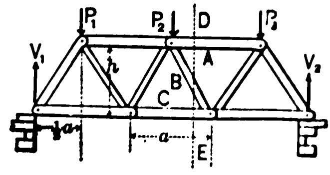


FIG. 38.

First, required the reactions of the supports V_1 and V_2; these and the loads are called the *external* forces. $\Sigma(Pa)$ about $O = 0$ gives

$$V_2 3a - P_1 \cdot \tfrac{1}{2}a - P_2 \cdot \tfrac{3}{2}a - P_3 \cdot \tfrac{5}{2}a = 0;$$

while $\Sigma(Pa)$ about $K = 0$ gives

$$- V_1 \cdot 3a + P_1 \cdot \tfrac{1}{2}a + P_2 \tfrac{3}{2}a + P_3 \tfrac{5}{2}a = 0;$$

$$\therefore V_1 = \tfrac{1}{6}[5P_1 + 3P_2 + P_3];$$

and $$V_2 = \tfrac{1}{6}[P_1 + 3P_2 + 5P_3].$$

Secondly, required the stress (thrust or pull, compression or tension) in each of the pieces A, B, and C cut by the imaginary line DE. The stresses in the pieces are called *internal* forces. These appear in a system of forces acting on a free body only when a portion of the truss or frame is conceived separated from the remainder in such a way as to expose an internal plane of one or more pieces. Consider as a free body the portion on the left of DE (that on the right would serve as well, but the pulls or thrusts in A, B, and C would be found to act in directions opposite to those they have on the other portion; see § 3). Fig. 39. The arrows (forces) A, B, and C are not pointed yet. They, with V_1, P_1, and P_2, form a system in equilibrium.

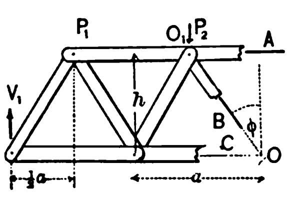


FIG. 39.

$\Sigma(Pa$ about $O = 0$ gives

$$(Ah) - V_1 2a + P_1 \cdot \tfrac{3}{2}a + P_2 \cdot \tfrac{1}{2}a = 0.$$

Therefore the moment $(Ah) = \tfrac{1}{2}a[4V_1 - 3P_1 - P_2]$, which is positive, since (from above) $4V_1$ is $> 3P_1 + P_2$. Hence A must point to the left, i.e., is a thrust or compression, and is $\dfrac{a}{2h}[4V_1 - 3P_1 - P_2]$.

Similarly, taking moments about O_1, the intersection of A and B, we have an equation in which the only unknown is C, viz., $(Ch) - V_1 \tfrac{3}{2}a + P_1 a = 0$. $\therefore (Ch) = \tfrac{1}{2}a[3V_1 - 2P_1]$,

a positive moment, since $3V_1$ is $>2P_1$; $\therefore C$ must point to the right, i.e., is a tension, and $= \frac{a}{2h}[3V_1 - 2P_1]$.

Finally, to obtain B, put Σ(vert. comps.) $= 0$; i.e. $(B \cos \varphi) + V_1 - P_1 - P_2 = 0$. $\therefore B \cos \varphi = P_1 + P_2 - V_1$; but (see foregoing value of V_1) we may write

$$V_1 = (P_1 + P_2) - (\tfrac{1}{6}P_1 + \tfrac{1}{2}P_2) + \tfrac{1}{6}P_2.$$

$\therefore B \cos \varphi$ will be $+$ (upward) or $-$ (downward), and B will be compression or tension, as $\frac{1}{6}P_2$ is $<$ or $>$ $[\frac{1}{6}P_1 + \frac{1}{2}P_2]$.

$$B = [P + P_1 - V_1] \div \cos \varphi = \frac{\sqrt{h^2 + \frac{1}{4}a^2}}{h}[P_1 + P_1 - V_1].$$

Problem 4.—Given the weight G_1 of rod, the weight G_2, and all the geometrical elements (the student will assume a

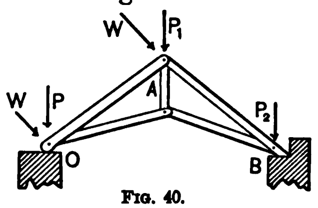


FIG. 40.

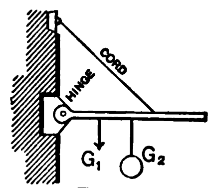


FIG. 41.

convenient notation); required the tension in the cord, and the amount and direction of pressure on hinge-pin.

Problem 5.—Roof-truss; pin-connection; all loads at joints; wind-pressures W_1 and W_2, normal to OA; required the three reactions or supporting forces (of the two horizontal surfaces and one vertical surface), and the stress in each piece. All geometrical elements are given; also P, P_1, P_2, W.

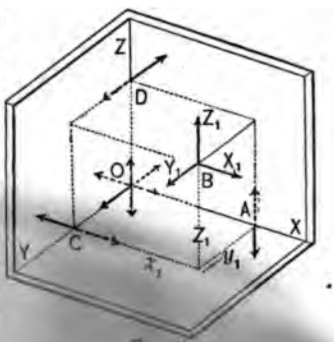


FIG. 42.

38. Composition of Non-concurrent Forces in Space.—Let P_1, P_2, etc., be the given forces, and x_1, y_1, z_1, x_2, y_2, z_2, etc., their points of application referred to an arbitrary origin and axes; α_1, β_1, γ_1, etc., the angles made by their lines of application with X, Y, and Z.

Considering the first force P_1, replace it by its three components parallel to the axes, $X_1 = P_1 \cos \alpha_1$; $Y_1 = P_1 \cos \beta_1$; and $Z_1 = P_1 \cos \gamma_1$ (P_1 itself is not shown in the figure). At O, and also at A, put a pair of equal and opposite forces, each equal and parallel to Z_1; Z_1 is now replaced by a single force Z_1 acting upward at the origin, and two couples, one in a plane parallel to YZ and having a moment $= - Z_1y_1$ (as we see it looking toward O from a remote point on the axis $+ X$), the other in a plane parallel to XZ and having a moment $= + Z_1x_1$ (seen from a remote point on the axis $+ Y$). Similarly at O and C put in pairs of forces equal and parallel to X_1, and we have X_1, at B, replaced by the single force X_1 at the origin, and the couples, one in a plane parallel to XY, and having a moment $+ X_1y_1$, seen from a remote point on the axis $+ Z$, the other in a plane parallel to XZ, and of a moment $= - X_1z_1$, seen from a remote point on the axis $+ Y$; and finally, by a similar device, Y_1 at B is replaced by a force Y_1 at the origin and two couples, parallel to the planes XY and YZ, and having moments $- Y_1x_1$ and $+ Z_1y_1$, respectively. (In Fig. 42 the single forces at the origin are broken lines, while the two forces constituting any one of the six couples may be recognized as being equal and parallel, of opposite directions, and both continuous, or both dotted.) We have, therefore, replaced the force P_1 by three forces X_1, Y_1, Z_1, at O, and six couples (shown more clearly in Fig. 43; the couples have been transferred to symmetrical positions). Combining each two couples whose axes are parallel to X, Y, or Z, they can be reduced to three, viz.,

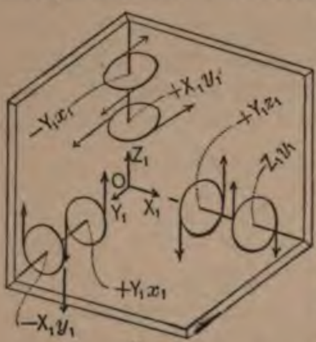


FIG. 43.

one with an X axis and a moment $= Y_1z_1 - Z_1y_1$;
one with a Y axis and a moment $= Z_1x_1 - X_1z_1$;
one with a Z axis and a moment $= X_1y_1 - Y_1x_1$.

Dealing with each of the other forces P_2, P_3, etc., in the same manner, the whole system may finally be replaced by three forces ΣX, ΣY, and ΣZ, at the origin and three couples whose moments are, respectively,

$$L = \Sigma(Yz - Zy) \text{ with its axis parallel to } X;$$
$$M = \Sigma(Zx - Xz) \text{ with its axis parallel to } Y;$$
$$N = \Sigma(Xy - Yx) \text{ with its axis parallel to } Z.$$

The "axes" of these couples, being parallel to the respective co-ordinate axes X, Y, and Z, and proportional to the moments L, M, and N, respectively, the axis of their resultant C, whose moment is G, must be the diagonal of a parallelopipedon constructed on the three component axes (proportional to) L, M, and N. Therefore, $G = \sqrt{L^2 + M^2 + N^2}$, while the resultant of ΣX, ΣY, and ΣZ is

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}$$

acting at the origin. If α, β, and γ are the direction-angles of R, we have $\cos\alpha = \frac{\Sigma X}{R}$, $\cos\beta = \frac{\Sigma Y}{R}$, and $\cos\gamma = \frac{\Sigma Z}{R}$; while if λ, μ, and ν are those of the *axis* of the couple C, we have $\cos\lambda = \frac{L}{G}$, $\cos\mu = \frac{M}{G}$, and $\cos\gamma = \frac{N}{G}$.

For equilibrium we have both $G = 0$ and $R = 0$; i.e., separately, *six conditions*, viz.,

$$\Sigma X = 0,\ \Sigma Y = 0,\ \Sigma Z = 0; \quad \text{and} \quad L = 0,\ M = 0,\ N = 0 \quad . \quad (1)$$

Now, noting that $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma(Xy - Yx) = 0$ are the conditions for equilibrium of the system of non-concurrent forces which would be formed by projecting each force of our actual system upon the plane XY, and similar relations for the planes YZ and XZ, we may restate equations (1) in another form, more serviceable in practical problems, viz.:

Note.—*If a system of non-concurrent forces in space is in equilibrium, the plane systems formed by projecting the given system upon each of three arbitrary co-ordinate planes will also be in equilibrium.* But we can obtain *only six independent*

equations in any case, available for six unknowns. If R alone $= 0$, we have the system equivalent to a couple C, whose moment $= G$; if G alone $= 0$, the system has a single resultant R applied at the origin. In general, neither R nor G being $= 0$, we cannot further combine R and C (as was done with non-concurrent forces in a plane) to produce a single resultant unless R and C are in the same plane; i.e., when the angle between R and the axis of C is $= 90°$. Call that angle θ. If, then, $\cos\theta = \cos\alpha\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu$ is $= 0 = \cos 90°$, we may combine R and C to produce a single resultant for the whole system; acting in a plane containing R and parallel to the plane of C in a direction parallel to R, at a perpendicular distance $c = \frac{G}{R}$ from the origin and $= R$ in intensity. The condition that a system of forces in space have a single resultant is, therefore, substituting the previously derived values of the cosines, $(\Sigma X).L + (\Sigma Y).M + (\Sigma Z).N = 0$.

This includes the cases when R is zero and when the system reduces to a couple.

To return to the general case, R and C not being in the same plane, the composition of forces in space cannot be further simplified. Still we can give any value we please to P, one of the forces of the couple C, calculate the corresponding arm $a = \frac{G}{P}$, then transfer C until one of the P's has the same point of application as R, and combine them by the parallelogram of forces. We thus have the whole system equivalent to two forces, viz., the second P, and the resultant of R and the first P. These two forces are not in the same plane, and therefore cannot be replaced by a single resultant.

39. Problem. (Non-concurrent forces in space.)—Given all geometrical elements (including α, β, γ, angles of P), also the weight of Q, and weight of apparatus G; A being a hinge whose pin is in the axis Y, O a ball-and-socket joint: required the *amount* of P (lbs.) to preserve equilibrium, also the pressures

(*amount* and *direction*) at A and O; no friction. Replace P by its X, Y, and Z components. The pressure at A will have

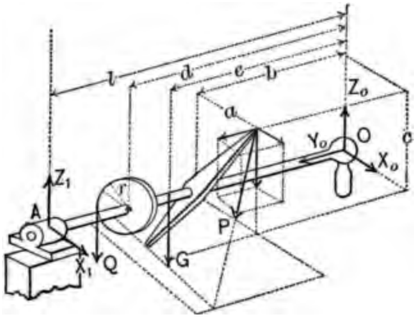


FIG. 44.

Z and X components; that at O, X, Y, and Z components. The body is now free, and there are six unknowns.

ΣX, ΣY, and ΣZ give, respectively,

$$P \cos \alpha + X_1 + X_0 = 0;$$
$$P \cos \beta + Y_0 = 0; \text{ and } Z_1 + Z_0 + Q + G - P \cos \gamma = 0.$$

As for moment-equations (see note in last paragraph), projecting the system upon YZ and putting $\Sigma(Pa)$ about $O = 0$, we have

$$-Z_1 l + Qd + Ge + (P \cos \gamma)b + (P \cos \beta)c = 0;$$

projecting it upon XZ, and putting $\Sigma(Pa)$ about $O = 0$, we have

$$Qr - (P \cos \alpha)c - (P \cos \gamma)a = 0;$$

projecting on XY, moments about O give

$$X_1 l + (P \cos \alpha)b - (P \cos \beta)a = 0.$$

From these six equations we may obtain the six unknowns, P, X_0, Y_0, Z_0, X_1, and Z_1. If for any one of these a negative result is obtained, it shows that its direction in Fig. 44 should be reversed.

CHAPTER IV.

STATICS OF FLEXIBLE CORDS.

40. Postulate and Principles.—The cords are perfectly flexible and inextensible. All problems will be restricted to one plane. Solutions of problems are based on three principles, viz.:

PRIN. I.—The strain on a cord at any point can act only along the cord, or along the tangent if it be curved.

PRIN. II.—We may apply to flexible cords in equilibrium all the conditions for the equilibrium of rigid bodies; since, if the system of cords became rigid, it would still, with greater reason, be in equilibrium.

PRIN. III.—The conditions of equilibrium cannot be applied, of course, unless the system can be considered a *free body*, which is allowable only when we conceive to be put in, at the points of support or fastening, the *reactions* (upon the cord) of those points and the supports removed. These reactions having been put in, then consider the case in Fig. 45 in one plane. If we take any point, p, *on the cord* as a centre of moments, knowing that the resultant R, of the forces P_1, P_2, and P_3, situated on *one side* of p, must act along the cord through p (by Prin. 1), therefore we have $P_1a_1 - P_2a_2 - P_3a_3 = R \times zero = 0$, and (equally well) $P_5a_5 - P_6a_6 - P_4a_4 = 0$. That is, *in a system of cords in equilibrium in a plane, if a centre of moments be taken on the cord, the algebraic sum of the mo-*

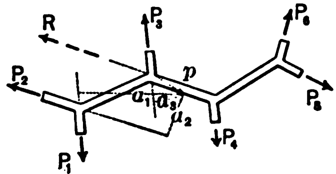


FIG. 45.

ments of those forces situated on one side (either) of this point will equal zero.

41. The Pulley.—A cord in equilibrium over a pulley whose axle is smooth has the same tension on both sides; for, Fig. 46,

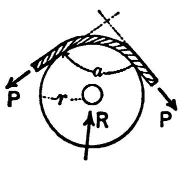


FIG. 46.

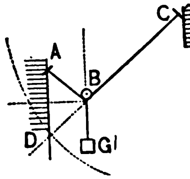


FIG. 47.

considering the pulley and its portion of cord free $\Sigma(Pa) = 0$ about the centre of axle gives $P'r = Pr$, i.e., $P' = P =$ tension in the cord. Hence the pressure R at the axle bisects the angle α, and therefore if a weighted pulley rides upon a cord ABC, Fig. 47, its position of equilibrium, B, may be found by cutting the vertical through A by an arc of radius $CD =$ length of cord, and centre at C, and drawing a horizontal through the middle of AD to cut CD in B. A smooth ring would serve as well as the pulley; this would be a *slip-knot.*

42. If three cords meet at a *fixed knot*, and are in equilibrium, the tension in any one is the equal and opposite of the resultant of those in the other two.

43. Tackle.—If a cord is continuous over a number of sheaves in blocks forming a tackle, neglecting the weight of the cord and blocks and friction of any sort, we may easily find the ratio between the cord-tension P and the weight to be sustained. E.g., Fig. 48, regarding all the straight cords as vertical and considering the block B free, we have, Fig. 49 (from $\Sigma Y = 0$), $4P - G = 0, \therefore P = \frac{G}{4}$. The stress on the support C will $= 5P$

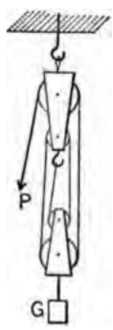


FIG. 48.

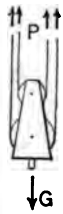


FIG. 49.

44. Weights Suspended by Fixed Knots.—Given all the geometrical elements in Fig. 50, and one weight, G_1; required the remaining weights and the forces H_0, V_0, H_n and V_n, at the points of support, that equilibrium may obtain. H_0 and V_0 are the horizontal and vertical components of the tension in the cord at O;

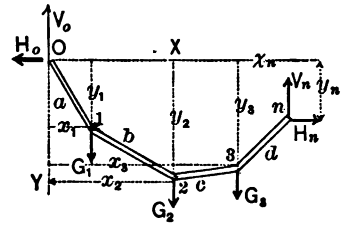


FIG. 50.

similarly H_n and V_n those at n. There are $n+2$ unknowns. From Prin. II we have $\Sigma X = 0$, and $\Sigma Y = 0$; i.e., $H_0 - H_n = 0$, and $[G_1 + G_2 + \ldots] - [V_0 + V_n] = 0$. While from Prin. III., taking the successive knots, 1, 2, etc., as centres of moments, we have

$$- V_0x_1 + H_0y_1 = 0,$$
$$- V_0x_2 + H_0y_2 + G_1(x_2 - x_1) = 0,$$
$$- V_0x_3 + H_0y_3 + G_1(x_3 - x_1) + G_2(x_3 - x_2) = 0,$$

etc., for n knots.

Thus we have $n+2$ independent equations, a sufficient number, and they are all of the first degree (with reference to the unknowns), and easily solved. As a special solution, we may, by § 42, resolve G_1 in the directions of the first and second cord-segments, and obtain their tensions by a parallelogram of forces; then at the second knot, knowing the tension in the second segment, we may find that in the third and G_2 in like manner, and so on. Of course H_0 and V_0 are components of the tension in the first segment, H_n and V_n of that in the last.

45. The converse of the problem in § 44, viz., given the weights G_1, etc., x_n and y_n, the lengths a, b, c, etc.; required H_0, V_0, H_n, V_n, and the co-ordinates x_1, y_1, x_2, y_2, etc., of the fixed knots when equilibrium exists, contains $2n+2$ unknowns. Statics furnishes $n+2$ equations (already given in § 44); while geometry gives the other n equations, one for each cord-segment, viz., $x_1^2 + y_1^2 = a^2$; $(x_2 - x_1)^2 + (y_2 - y_1)^2 = b^2$; etc.

However, most of these $2n+2$ equations are of the second degree; hence in the general case they cannot be solved.

46. Loaded Cord as Parabola.—If the weights are equal and infinitely small, and are intended to be uniformly spaced *along the horizontal*, when equilibrium obtains, the cord having no weight, it will form a parabola. Let q = weight of loads per horizontal linear unit, O be the vertex of the curve in which the cord hangs, and m any point. We may consider the portion Om as a **free body**, if the reactions of the contiguous portions of the cord are put in, H_0 and T, and these (from Prin. I.) must act along the tangents to the curve at O and m, respectively; i.e., H_0 is horizontal, and T makes some angle φ (whose tangent $= \frac{dy}{dx}$, etc.) with the axis X. Applying Prin. II.,

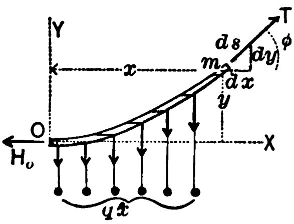


FIG. 51.

$$\Sigma X = 0 \text{ gives } T\cos\varphi - H = 0;\ \text{i.e., } T\frac{dx}{ds} = H_0; \quad . \quad . \quad (1)$$

$$\Sigma Y = 0 \text{ gives } T\sin\varphi - qx = 0;\ \text{i.e., } T\frac{dy}{ds} = qx. \quad . \quad . \quad (2)$$

Dividing (2) by (1), member by member, we have $\frac{dy}{dx} = \frac{qx}{H_0}$;

$\therefore dy = \frac{q}{H_0}xdx$, the differential equation of the curve; $y = \frac{q}{H_0}\int_0^x xdx = \frac{q}{H_0}\cdot\frac{x^2}{2}$; or $x^2 = \frac{2H_0}{q}y$, the equation of a **parabola** whose vertex is at O and axis vertical.

NOTE.—The same result, $\frac{dy}{dx} = \frac{qx}{H_0}$, may be obtained by considering that we have here (Prin. II.) a *free rigid body* acted on by three forces, T, H_0, and $R = qx$, acting vertically through the middle of the abscissa x; the resultant of H_0 and R must be equal and opposite to T, Fig. 52. $\therefore \tan\varphi = \frac{R}{H_0}$, or Evidently also the tangent-line bisect scissa x.

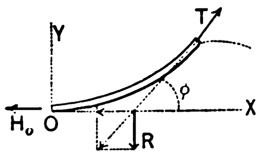


FIG. 52.

47. Problem under § 46. [Case of a suspension-bridge in which the suspension-rods are vertical, the weight of roadway is uniform per horizontal foot, and large compared with that of the cable and rods. Here the roadway is the only load: it is generally furnished with a stiffening truss to avoid deformation under passing loads.]—Given the span $= 2b$, Fig. 53, the deflection $= a$, and the rate of loading $= q$ lbs. per horizontal foot; required the tension in the cable at O, also at m'; and the length of cable needed. From the equation of the parabola $qx^2 = 2H_0y$, putting $x = b$ and $y = a$, we have $H_0 = qb^2 \div 2a =$ the tension at O. From $\Sigma Y = 0$ we have $V_1 = qb$, while $\Sigma X = 0$ gives

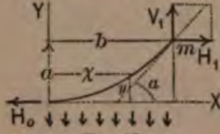


FIG. 53.

$H_1 = H_0$; $\therefore$ the tension at $m' = \sqrt{H_1^2 + V_1^2} = \frac{1}{2a}[qb\sqrt{4a^2 + b^2}]$.

The semi-length, Om', of cable (from p. 88, Todhunter's Integral Calculus) is (letting n denote $H_0 \div 2q$)

$$Om' = \sqrt{na + a^2} + n \,.\, \log_e [(\sqrt{a} + \sqrt{n + a}) \div \sqrt{n}].$$

48. The Catenary.—A flexible, inextensible cord or chain, of uniform weight per unit of length, hung at two points, and supporting *its own weight alone*, forms a curve called the **catenary.** Let the tension H_0 at the lowest point or vertex be represented (for algebraic convenience) by the weight of an imaginary length, c, of similar cord weighing q lbs. per unit of length, i.e., $H_0 = qc$; an actual portion of the cord, of length s, weighs qs lbs. Fig. 54 shows as *free* and in equilibrium a portion of the curve of any length s, reckoning from O the vertex. Required the equation of the curve. The load is uniformly spaced *along the curve*, and not horizontally, as in §§ 46 and 47.

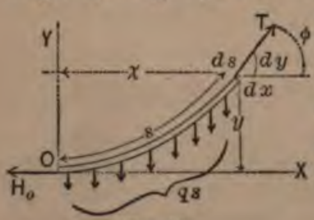


FIG. 54.

$\Sigma Y = 0$ gives $T\dfrac{dy}{ds} = qs$; while $\Sigma X = 0$ gives $T\dfrac{dx}{ds} = qc$. Hence, by division, $cdy = sdx$, and squaring $c^2dy^2 = s^2dx^2$. . . . . (1)

Put $dy^2 = ds^2 - dx^2$, and we have, after solving for dx,

$$dx = \frac{c\,ds}{\sqrt{s^2 + c^2}}. \quad \therefore x = c\int_0^s \frac{ds}{\sqrt{s^2 + c^2}} = c\Big[_0^s \log_\epsilon (s + \sqrt{s^2 + c^2}),$$

and $$x = c \,.\, \log_\epsilon [(s + \sqrt{s^2 + c^2}) \div c], \quad . \quad . \quad . \quad (2)$$

a relation between the horizontal abscissa and length of curve.

Again, in eq. (1) put $dx^2 = ds^2 - dy^2$, and solve for dy.

This gives $dy = \dfrac{s\,ds}{\sqrt{c^2 + s^2}} = \dfrac{1}{2} \cdot \dfrac{d(c^2 + s^2)}{(c^2 + s^2)^{\frac{1}{2}}}$. Therefore

$y = \frac{1}{2}\int_0^s (c^2 + s^2)^{-\frac{1}{2}} d(c^2 + s^2) = \frac{1}{2}\Big[_0^s 2(c^2 + s^2)^{\frac{1}{2}}$, and finally

$$y = \sqrt{s^2 + c^2} - c. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Clearing of radicals and solving for c, we have

$$c = (s^2 - y^2) \div 2y. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Example.—A 40-foot chain weighs 240 lbs., and is so hung from two points at the same level that the deflection is 10 feet. Here, for $s = 20$ ft., $y = 10$; hence eq. (4) gives the *parameter*, $c = (400 - 100) \div 20 = 15$ feet. $q = 240 \div 40 = 6$ lbs. per foot. $\therefore$ the tension at the middle is $H_0 = qc = 6 \times 15 = 90$ lbs.; while the greatest tension is at either support and $= \sqrt{90^2 + 120^2} = 150$ lbs.

Knowing $c = 15$ feet, and putting $s = 20$ feet $=$ half length of chain, we may compute the corresponding value of x from eq. (2); this will be the half-span [$\log_\epsilon m = 2.30258 \times$ (common log m)]. To derive s in terms of x, transform eq. (2) in the sense in which $n = \log_\epsilon m$ may be transformed into $\epsilon^n = m$, clear of radicals, and solve for s, which gives

$$s = \tfrac{1}{2}c\left[\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Again, eliminate s from (2) by substitution from (3), transform as above, clear of radicals, and solve for $y + c$, whence

$$y + c = \tfrac{1}{2}c\left[\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which is the equation of a catenary with axes as in Fig. 54. If the horizontal axis be taken, a distance $= c$ below the vertex, the new ordinate $y' = y + c$, while x remains the same; the last equation is simplified.

If the span and length of chain are given, or if the span and deflection are given, c can be determined from (4) or (5) only by successive assumptions and approximations.

PART II.—DYNAMICS.

CHAPTER I.

RECTILINEAR MOTION OF A MATERIAL POINT.

49. Uniform Motion implies that the moving point passes over equal distances in equal times; **variable motion**, that unequal distances are passed over in equal times. In uniform motion the distance passed over in a unit of time, as one second, is called the **velocity** ($= v$), which may also be obtained by dividing the length of *any portion* ($= s$) of the path by the time ($= t$) taken to describe that portion, however small or great; in variable motion, however, the velocity varies from point to point, its value at any point being expressed as the quotient of ds (an infinitely small distance containing the given point) by dt (the infinitely small portion of time in which ds is described).

49*a*. By **acceleration** is meant the rate at which the velocity of a variable motion is changing at any point, and may be a *uniform acceleration*, in which case it equals the total change of velocity between any two points, however far apart, divided by the time of passage; or a *variable acceleration*, having a different value at every point, this value then being obtained by dividing the velocity-increment, dv, or gain of velocity in passing from the given point to one infinitely near to it, by dt, the time occupied in acquiring the gain. (Acceleration must be understood in an algebraic sense, a negative acceleration implying a decreasing velocity, or else that the velocity in a negative direction is increasing.) The foregoing applies to motion in a path or line of any form whatever, the distances mentioned being portions of the path, and therefore measured along the path.

50. Rectilinear Motion, or motion in a straight line.—The general relations of the quantities involved may be thus stated (see Fig. 55): Let v = velocity of the body at any instant; then dv = gain of velocity in an instant of time dt. Let t = time elapsed since the body left a given fixed point,

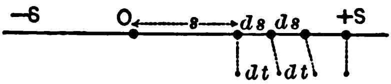


FIG. 55.

which will be taken as an origin, O. Let s = distance (+ or −) of the body, at any instant, from the origin O; then ds = distance traversed in a time dt. Let p = acceleration = *rate at which v is increasing* at any instant. All these may be variable; and t is taken as the **independent variable**, i.e., time is conceived to elapse by *equal* small increments, each = dt; hence two consecutive ds's will not in general be equal, their difference being called d^2s. Evidently d^2t is = zero, i.e., dt is constant.

Since $\frac{1}{dt}$ = number of instants in one second, the velocity at any instant (i.e., the distance which *would* be described *at that rate* in one second) is $v = ds \cdot \frac{1}{dt}$; $\therefore v = \frac{ds}{dt}$. . . . . (I.)

Similarly, $p = dv \cdot \frac{1}{dt}$, and $\left(\text{since } dv = d\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt}\right)$.

$$\therefore p = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \quad . \; . \; . \; . \; . \; \text{(II.)}$$

Eliminating dt, we have also $vdv = pds$. . . . . . (III.)

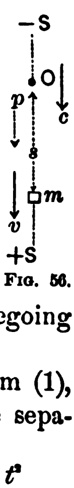
These are the fundamental differential formulæ of rectilinear motion (for curvilinear motion we have these and some in addition) as far as kinematics, i.e., as far as space and time, is concerned. The consideration of the mass of the material point and the forces acting upon it will give still another relation (see § 55).

51. Rectilinear Motion due to Gravity.—If a material point fall freely in vacuo, no initial direction other than vertical having been given to its motion, many experiments have

shown that this is a uniformly accelerated rectilinear motion in a vertical line having an acceleration (called the *acceleration of gravity*) equal to 32.2 feet per square second, or 9.81 metres per square second; i.e., the velocity increases at this constant rate in a downward direction, or decreases in an upward direction.

[NOTE.—By "square second" it is meant to lay stress on the fact that an acceleration (being $= d^2s \div dt^2$) is in quality equal to one dimension of length divided by two dimensions of time. E.g., if instead of using the foot and second as units of space and time we use the foot and the minute, g will $= 32.2 \times 3600$; whereas a velocity of say six feet per second would $= 6 \times 60$ feet per minute. The value of $g = 32.2$ implies the units foot and second, and is sufficiently exact for practical purposes.]

52. Free Fall in Vacuo.—Fig. 56. Let the body start at O with an initial downward velocity $= c$. The acceleration is constant and $= +g$. Reckoning both time and distance (+ downwards) from O, required the values of the variables s and v after any time t. From eq. (II.), § 50, we have $+g = dv \div dt$; $\therefore dv = gdt$, in which the two variables are separated.



Hence $\int_c^v dv = g\int_0^t dt$; i.e., $\Big[_c^v v = g\Big[_0^t t$; or $v - c = gt - 0$; and finally, $v = c + gt$. . . . . (1) FIG. 56.

(Notice the correspondence of the limits in the foregoing operation; when $t = 0$, $v = +c$.)

From eq. (I.), § 50, $v = ds \div dt$; $\therefore$ substituting from (1), $ds = (c + gt)dt$, in which the two variables s and t are separated.

$$\therefore \int_0^s ds = c\int_0^t dt + g\int_0^t tdt; \text{ i.e., } \Big[_0^s s = c\Big[_0^t t + g\Big[_0^t \frac{t^2}{2},$$

or $$s = ct + \tfrac{1}{2}gt^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, eq. (III.), § 50, $vdv = gds$, in which the variables v and s are already separated.

$$\therefore \int_c^v vdv = g\int_0^s ds; \text{ or } \Big[_c^v \tfrac{1}{2}v^2 = g\Big[_0^s s; \text{ i.e., } \tfrac{1}{2}(v^2 - c^2) = gs,$$

or $$s = \frac{v^2 - c^2}{2g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the initial velocity = zero, i.e., if the body falls from rest, eq. (3) gives $s = \frac{v^2}{2g}$ and $v = \sqrt{2gh}$. [From the frequent recurrence of these forms, especially in hydraulics, $\frac{v^2}{2g}$ is called the "height due to the velocity v," i.e., the vertical height through which the body must fall from rest to acquire the velocity v; while, conversely, $\sqrt{2gh}$ is called the velocity due to the height or head h.]

By eliminating g between (1) and (3), we may derive another formula between three variables, s, v, and t, viz.,

$$s = \tfrac{1}{2}(c + v)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

53. Upward Throw.—If the initial velocity were in an upward direction in Fig. 56 we might call it $-c$, and introduce it with a negative sign in equations (1) to (4), just derived; but for variety let us call the upward direction $+$, in which case an upward initial velocity would $= +c$, while the acceleration $= -g$, constant, as before. (The motion is supposed confined within such a small range that g does not sensibly vary.) Fig. 57. From $p = dv \div dt$ we have $dv = -gdt$ and

$$\int_c^v dv = -g\int_0^t dt; \therefore v - c = -gt; \text{ or } v = c - gt. \quad (1)a$$

From $v = ds \div dt$, $ds = cdt - gtdt$,

$$\text{i.e., } \int_0^s ds = c\int_0^t dt - g\int_0^t tdt; \text{ or } s = ct - \tfrac{1}{2}gt^2. \quad (2)a$$

$vdv = pds$ gives $\int_c^v vdv = -g\int_0^s ds$, whence

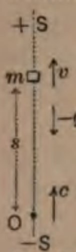


FIG. 57.

$$\tfrac{1}{2}(v^2 - c^2) = -gs, \text{ or finally, } s = \frac{c^2 - v^2}{2g}. \quad . \quad (3)a$$

And by eliminating g from (1)a and (3)a,

$$s = \tfrac{1}{2}(c + v)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)a$$

The following is now easily verified from these equations: the body passes the origin again ($s = 0$) with a velocity $= -c$, after a lapse of time $= 2c \div$ [illegible] body comes to rest (

an instant) (put $v = 0$) after a time $= c \div g$, and at a distance $s = c^2 \div 2g$ ("height due to velocity c") from O. For $t > c \div g$, v is negative, showing a downward motion; for $t > 2c \div g$, s is negative, i.e., the body is below the starting-point while the rate of change of v is constant and $= -g$ at all points.

54. Newton's Laws.—As showing the relations existing in general between the motion of a material point and the actions (forces) of other bodies upon it, experience furnishes the following three laws or statements as a basis for dynamics:

(1) A material point under no forces, or under balanced forces, remains in a state of rest or of uniform motion in a right line. (This property is often called *Inertia.*)

(2) If the forces acting on a material point are unbalanced, an acceleration of motion is produced, proportional to the resultant force and in its direction.

(3) Every action (force) of one body on another is always accompanied by an equal, opposite, and simultaneous reaction. (This was interpreted in § 3.)

As all bodies are made up of material points, the results obtained in Dynamics of a Material Point serve as a basis for the Dynamics of a Rigid Body, of Liquids, and of Gases.

55. Mass.—If a body is to continue moving in a right line, the resultant force P at all instants must be directed along that line (otherwise it would have a component deflecting the body from its straight course).

In accordance with Newton's second law, denoting by p the acceleration produced by the resultant force (G being the body's weight), we must have the proportion $P : G :: p : g$; i.e.,

$$P = \frac{G}{g} p \text{ ......, or } P = Mp. \quad . \quad . \quad \text{(IV.)}$$

E[illegible] and (I. [illegible] III.) of § 50 are the fundamental [illegible]nce the quotient $G \div g$ is invaria-

ble, wherever the body be moved on the earth's surface (G and g changing in the same ratio), it will be used as the measure of the mass M or quantity of matter in the body. In this way it will frequently happen that the quantities G and g will appear in problems where the weight of the body, i.e., the force of the earth's attraction upon it, and the acceleration of gravity have no direct connection with the circumstances. No name will be given to the unit of mass, it being always understood that the fraction $G \div g$ will be put for M before any numerical substitution is made. From (IV.) we have, in words,

$$\begin{cases} \qquad \textit{accelerating force} = \textit{mass} \times \textit{acceleration}; \\ \text{also,} \quad \textit{acceleration} = \textit{accelerating force} \div \textit{mass}. \end{cases}$$

56. Uniformly Accelerated Motion.—If the resultant force is **constant** as time elapses, the acceleration must be constant (from eq. (IV.), since of course M is constant) and $= P \div M$. The motion therefore will be uniformly accelerated, and we have only to substitute $+p$ (constant) for g in eqs. (1) to (4) of § 52 for the equations of this motion, the initial velocity being $= c$ (in the line of the force).

$$v = c + pt \quad . \quad . \quad . \quad (1); \qquad s = ct + \tfrac{1}{2}pt^2; \quad . \quad . \quad . \quad (2)$$

$$s = \frac{(v^2 - c^2)}{2p}; \quad . \quad . \quad . \quad (3), \quad \text{and } s = \tfrac{1}{2}(c + v)t \quad . \quad . \quad . \quad (4)$$

If the force is in a negative direction, the acceleration will be negative, and may be called a *retardation;* the initial velocity should be made negative if its direction requires it.

57. Examples of Unif. Acc. Motion.—*Example* 1. Fig. 58. A small block whose weight is $\frac{1}{2}$ lb. has already described a distance $Ao = 48$ inches over a smooth portion of a horizontal table in two seconds; at O it encounters a rough portion, and a consequent constant friction of 2 oz. Required the distance described beyond O, and the time occupied in coming to rest. Since we shall use 32.2 for g, times must be in seconds, and distances in feet; as to the unit

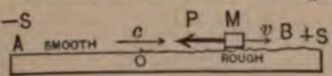


FIG. 58.

of force, as that is still arbitrary, say ounces. Since AO was smooth, it must have been described with a uniform motion (the resistance of the air being neglected); hence with a velocity $= 4$ ft. $\div$ 2 sec. $= 2$ ft. per sec. The initial velocity for the retarded motion, then, is $c = +2$ at O. At any point beyond O the acceleration $=$ force $\div$ mass $= (-2$ oz.$) \div (8$ oz. $\div 32.2) = -8.05$ ft. per square second, i.e., $p = -8.05 =$ constant; hence the motion is uniformly accelerated (retarded here), and we may use the formulæ of § 56 with $c = +2$, $p = -8.05$. At the end of the motion v must be zero, and the corresponding values of s and t may be found by putting $v = 0$ in equations (3) and (1), and solving for s and t respectively: thus from (3), $s = (-4) \div (-8.05)$, i.e., $s = 0.497 +$, which must be feet; while from (1), $t = (-2) \div (-8.05) = 0.248 +$, which must be seconds.

Example 2. (Algebraic.)—Fig. 59. The two masses $M_1 = G_1 \div g$ and $M = G \div g$ are connected by a flexible, inextensible cord. Table smooth. Required the acceleration common to the two rectilinear motions, and the tension in the string S,

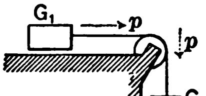


FIG. 59.

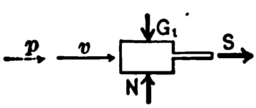


FIG. 60.

there being no friction under G_1, none at the pulley, and *no mass* in the latter or in the cord. At any instant of the motion consider G_1 free (Fig. 60), N being the pressure of the table against G_1. Since the motion is in a horizontal right line Σ(vert. compons.)$= 0$, i.e., $N - G_1 = 0$, which determines N. S, the only horizontal force (and resultant of all the forces) $= M_1 p$, i.e.,

$$S = G_1 p \div g. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At the same instant of the motion consider G free (Fig. 61); the tension in the cord is the same value as above $= S$. The accelerating force is $G - S$, and

$$\therefore = \text{mass} \times \text{acc., or } G - S = (G \div g)p. \quad . \quad (2)$$

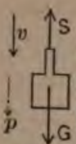


FIG. 61.

From equations (1) and (2) we obtain $p = (Gg) \div (G + G_1) =$ a constant; hence each motion is *uniformly accelerated*, and we may employ equations (1) to (4) of § 56 to find the velocity and distance from the starting-points, at the end of any assigned time t, or *vice versâ*. The initial velocity must be known, and may be zero. Also, from (1) and (2) of this article,

$$S = (GG_1) \div (G + G_1) = \text{constant.}$$

Example 3.—A body of $2\frac{3}{4}$ (short) tons weight is acted on during $\frac{1}{2}$ minute by a constant force P. It had previously described $316\frac{2}{3}$ yards in 180 seconds under no force; and subsequently, under no force, describes 9900 inches in $\frac{1}{40}$ of an hour. Required the value of P. Ans. $P = 22.1$ lbs.

Example 4.—A mass of 1 ton having an initial velocity of 48 inches per second, is acted on for $\frac{1}{4}$ minute by a force of 400 avoirdupois ounces. Required the final velocity.

Ans. 10.037 ft. per sec.

Example 5.—Initial velocity, 60 feet per second; mass weighs 0.30 of a ton. A resistance of $112\frac{1}{2}$ lbs. retards it for $\frac{2}{15}$ of a minute. Required the distance passed over during this time.

Ans. 286.8 feet.

Example 6.—Required the time in which a force of 600 avoirdupois ounces will increase the velocity of a mass weighing $1\frac{1}{2}$ tons from 480 feet per minute to 240 inches per second.

Ans. 30 seconds.

Example 7.—What distance is passed over by a mass of (0.6) tons weight during the overcoming of a constant resistance (friction), if its velocity, initially 144 inches per sec., is reduced to zero in 8 seconds. Required, also, the friction.

Ans. 48 ft. and 55 lbs.

Example 8.—Before the action of a force (value required) a body of 11 tons had described uniformly 950 ft. in 12 minutes. Afterwards it describes 1650 feet uniformly in 180 seconds. The force acts 30 seconds. $P = ?$ Ans. $P = 178$ lbs.

58. Graphic Representations. Unif. Acc. Motion.—With the initial velocity = 0, the equations of § 56 become

$$v = pt, \quad . \quad . \quad . \quad . \quad . \quad (1) \qquad s = \tfrac{1}{2}pt^2, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$s = v^2 \div 2p, \quad . \quad . \quad . \quad (3) \quad \text{and} \quad s = \tfrac{1}{2}vt. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Eqs. (1), (2), and (3) contain each two variables, which may graphically be laid off to scale as co-ordinates and thus give a curve corresponding to the equation. Thus, Fig. 62, in (I.), we

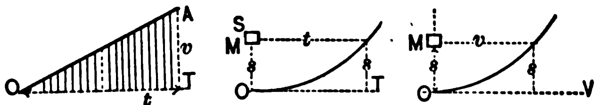


FIG. 62.

have a right line representing eq. (I.); in (II.), a parabola with axis parallel to s, and vertex at the origin for eq. (2); also a parabola similarly situated for eq. (3). Eq. (4) contains three variables, s, v, and t. This relation can be shown in (I.), s being represented by the *area* of the shaded triangle = $\frac{1}{2}vt$. (II.) and (III.) have this advantage, that the axis OS may be made the actual path of the body. [Let the student determine how the origin shall be moved in each case to meet the supposition of an initial velocity = + c or − c.]

59. Variably Accelerated Motions.—We here restate the equations

$$v = \frac{ds}{dt} \quad . \quad . \quad (\text{I.}); \quad p = \frac{dv}{dt} = \frac{d^2s}{dt^2} \quad . \quad . \quad (\text{II.}); \quad vdv = pds \quad . \quad . \quad (\text{III.});$$

and resultant force

$$= P = Mp, \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{IV.});$$

which are the only ones for *general use* in rectilinear motion.

PROBLEM 1.—In pulling a mass M along a smooth, horizontal table, by a horizontal cord, the tension is so varied that $s = 4t^3$ (*not a* [redacted] *equation;* the units are, say, the foot and [illegible] by what law the tension varies.

From (I.) $v = \frac{ds}{dt} = \frac{d(4t^3)}{dt} = 12t^2$; from (II.), $p = \frac{d(12t^2)}{dt} = 12t$; and (IV.) the tension $= P = Mp = 24Mt$, i.e., varies directly as the time.

PROBLEM 2. "Harmonic Motion," Fig. 63.—A small block

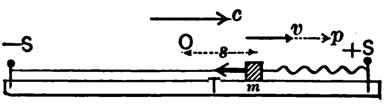


FIG. 63

on a smooth horizontal table is attached to two horizontal elastic cords (and they to pegs) in such a way that when the block is at O, each cord is straight but not tense; in any other position, as m, one cord is tense, the other slack. The cords are alike in every respect, and, like most springs, the tension varies directly with the elongation ($= s$ in figure). If for an elongation s_1 the tension is T_1, then for any elongation s it is $T = T_1 s \div s_1$. The acceleration at any point m, then, is $p = -(T \div M) = -(T_1 s \div M s_1)$, which for brevity put $p = -as$, a being a constant. Required the equations of motion, the initial velocity being $= +c$, at O. From eq. (III.)

$$vdv = -asds; \therefore \int_c^v vdv = -a\int_0^s sds,$$

$$\text{i.e., } \tfrac{1}{2}(v^2 - c^2) = -\tfrac{1}{2}as^2; \quad \text{or, } v^2 = c^2 - as^2. \quad . \quad (1)$$

From (I.), $dt = ds \div v$; hence from (1),

$$\int_0^t dt = \int_0^s [ds \div \sqrt{c^2 - as^2}],$$

or

$$t = \frac{1}{\sqrt{a}}\int_0^s \frac{d[s\sqrt{a} \div c]}{\sqrt{1-(s\sqrt{a} \div c)^2}} = \frac{1}{\sqrt{a}}\Big[\sin^{-1}\Big(\frac{s\sqrt{a}}{c}\Big)\Big]_0^s$$

$$= \frac{1}{\sqrt{a}}\sin^{-1}\Big(\frac{s\sqrt{a}}{c}\Big). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Inverting (2), we have $s = (c \div \sqrt{a}), \sin (t\sqrt{a}), \quad . \quad . \quad .$ (3)

Again, by differentiating (3), see (I.), $v = c \cos (t\sqrt{a})$ (4)

Differentiating (4), see (II.), $p = -c\sqrt{a} \sin (t\sqrt{a}). \quad . \quad .$ (5)

These are the relations required, but the peculiar property of the motion is made apparent by inquiring the time of passing from O to a state of rest; i.e., put $v = 0$ in equation (4), we obtain $t = \frac{1}{2}\pi \div \sqrt{a}$, or $\frac{3}{2}\pi \div \sqrt{a}$, or $\frac{5}{2}\pi \div \sqrt{a}$, and so on, while the corresponding values of s (from equation (3)), are $+(c \div \sqrt{a}), -(c \div \sqrt{a}), +(c \div \sqrt{a})$, and so on. This shows that the body vibrates equally on both sides of O in a cycle or period whose duration $= 2\pi \div \sqrt{a}$, and is *independent of the initial velocity given it at O.* Each time it passes O the velocity is either $+c$, or $-c$, the acceleration $= 0$, and the time since the start is $= 2n\pi \div \sqrt{a}$, in which n is any whole number. At the extreme point $p = \mp c\sqrt{a}$, from eq. (5). If then a different amplitude be given to the oscillation by changing c, the duration of the period is still the same, i.e., the vibration is *isochronal.* The motion of an ordinary pendulum is nearly, that of a cycloidal pendulum exactly, harmonic.

If the crank-pin of a reciprocating engine moves uniformly in its circular path, the piston would have a harmonic motion if the connecting-rod were infinitely long, or if the design in

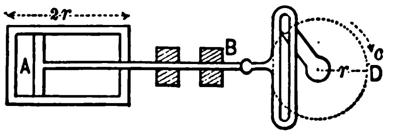


FIG. 64.

Fig. 64 were used. (Let the student prove this from eq. (3).) Let $2r$ = length of stroke, and c = the uniform velocity of the crank-pin, and M = mass of the piston and rod AB. Then the velocity of M at mid-stroke must $= c$, at the dead-points, zero; its a[illegible]ation at mid-stroke zero; at the dead-points [illegible]e ac[illegible]d, $s = r$ [illegible] $\div \sqrt{a}$ (from eq. (3)); $\therefore \sqrt{a}$ [illegible] point (the maximum acc.)

$= c^2 \div r$. Hence on account of the acceleration (or retardation) of M in the neighborhood of a dead-point a pressure will be exerted on the crank-pin, equal to mass $\times$ acc. $= Mc^2 \div r$ at those points, independently of the force transmitted due to steam-pressure on the piston-head, and makes the resultant pressure on the pin at C smaller, and at D larger than it would be if the "*inertia*" of the piston and rod were not thus taken into account. We may prove this also by the free-body method, considering *AB free* immediately after passing the dead-point

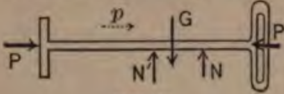


FIG. 65.

C, neglecting all friction. See Fig. 65. The forces acting are: G, the weight; N, the pressures of the guides; P, the known effective steam-pressure on piston-head; and P', the unknown pressure of crank-pin on side of slot. There is no change of motion vertically; $\therefore N' + N - G = 0$, and the resultant force is $P - P'$ $=$ mass $\times$ accel. $= Mc^2 \div r$, hence $P' = P - Mc^2 \div r$. Similarly at the other dead-point we would obtain $P' = P + Mc^2 \div r$. In high-speed engines with heavy pistons, etc., $Mc^2 \div r$ is no small item.

PROBLEM 3.—Supposing the earth at rest and *the resistance of the air to be null*, a body is given an initial upward vertical velocity $= c$. Required the velocity at any distance s from the centre of the earth, whose attraction varies inversely as the square of the distance s.

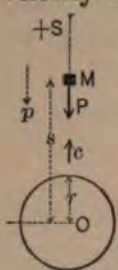


FIG. 66.

See Fig. 66.—The attraction on the body at the surface of the earth where $s = r$, the radius, is its weight G; at any point m it will be $P = G(r^2 \div s^2)$, while its mass $= G \div g$.

Hence the acceleration at $m = p = (-P) \div M$ $= -g(r^2 \div s^2)$. Take equation III., $vdv = pds$, and we have

$$vdv = -gr^2 s^{-2} ds; \therefore$$

$$\int_c^v vdv = -gr^2 \int_r^s s^{-2} ds; \text{ or, } \Big[_c^v \tfrac{1}{2}v^2 = -gr^2 \Big[_r^s -\frac{1}{s},$$

$$\text{i.e., } \tfrac{1}{2}(v^2 - c^2) = -gr^2\left(\frac{1}{r} - \frac{1}{s}\right). \quad . \quad . \quad . \quad (1)$$

Evidently v decreases, as it should. Now inquire how small a value c may have that the body shall *never return;* i.e., that v shall not $= 0$ until $s = \infty$. Put $v = 0$ and $s = \infty$ in (1) and solve for c; and we have

$$c = \sqrt{2gr} = \sqrt{2 \times 32.2 \times 21000000},$$

= about 36800 ft. per sec. or nearly 7 miles per sec. Conversely, if a body be allowed to fall, from rest, toward the earth, the velocity with which it would strike the surface would be less than seven miles per second through whatever distance it may have fallen.

If a body were allowed to fall through a straight opening in the earth passing through the centre, the motion would be harmonic, since the attraction and consequent acceleration now vary directly with the distance from the centre. See Prob. 2. This supposes the earth homogeneous.

PROBLEM 4.—Steam working expansively and raising a weight. Fig. 67.—A piston works without friction in a vertical cylinder. Let S = total steam-pressure on the under side of the piston; the weight G, of the mass $G \div g$ (which includes the piston itself) and an atmospheric pressure $= A$, constitute a constant back-pressure. Through the portion $OB = s_1$, of the stroke, S is constant $= S_1$, while beyond B, boiler communication being "cut off," S diminishes with Boyle's law, i.e., in this case, for any point m above B, we have, neglecting the "clearance", F being the cross-section of the cylinder,

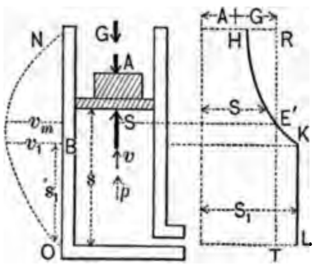


FIG. 67.

$$S : S_1 :: Fs_1 : Fs; \quad \text{or} \quad S = S_1 s_1 \div s.$$

Full length of stroke $= ON = s_n$. Given, then, the forces S_1 and A, the distances s_1 and s_n, and the velocities at O and at N both $= 0$ (i.e., the mass $M = G \div g$ is to start from rest at O, and to *come to rest at N*), required the proper weight G to

fulfil these conditions, S varying as already stated. The acceleration at any point will be

$$p = [S - A - G] \div M. \quad . \quad . \quad . \quad . \quad (1)$$

Hence (eq. III.) $Mvdv = [S - A - G]ds$, and $\therefore$ for the whole stroke

$$M\int_0^0 vdv = \int_0^N [S - A - G]ds; \text{ i.e.,}$$

$$0 = S_1\int_0^{s_1} ds + S_1s_1\int_{s_1}^{s_n}\frac{ds}{s} - A\int_0^{s_n} ds - G\int_0^{s_n} ds,$$

or

$$S_1s_1\left[1 + \log_e \frac{s_n}{s_1}\right] = As_n + Gs_n. \quad . \quad . \quad . \quad (2)$$

Since $S = S_1 =$ constant, from O to B, and variable, $=$ $S_1s_1 \div s$, from B to N, we have had to write the summation $\int_0^N Sds$ in two parts.

From (2), G becomes known, and $\therefore$ M also ($= G \div g$).

Required, further, the time occupied in this upward stroke. From O to B (the point of cut-off) the motion is uniformly accelerated, since p is constant (S being $= S_1$ is eq. (1)), with the initial velocity zero; hence, from eq. (3), § 56, the velocity at $B = v_1 = \sqrt{2\,[S_1 - A - G]s_1 \div M}$ is known; $\therefore$ the time $t_1 = 2s_1 \div v_1$ becomes known (eq. (4), § 56) of describing OB. At any point beyond B the velocity v may be obtained thus: From (III.) $vdv = pds$, and eq. (1) we have, summing between B and any point above,

$$M\int_{v_1}^{v} vdv = S_1s_1\int_{s_1}\frac{ds}{s} - (A + G)\int_{s_1}^{s} ds; \text{ i.e.,}$$

$$\frac{G}{g}\frac{(v^2 - v_1^2)}{2} = S_1s_1 \log_e \frac{s}{s_1} - (A + G)(s - s_1).$$

This gives the relation between the two variables v and s anywhere between B and N; if we solve for v and insert its value in $dt = ds \div v$, we shall have $dt =$ a function of s and ds, which is not integrable. Hence we may resort to approxi-

mate methods for the time from B to N. Divide the space BN into an uneven number of equal parts, say five; the distances of the points of division from O will be s_1, s_2, s_3, s_4, s_5, and s_n. For these values of S compute (from above equation) v_1 (already known), v_2, v_3, v_4, v_5, and v_n (known to be zero). To the first four spaces apply Simpson's Rule, and we have the time from B to the end of s_5,

$$\Big[_1^5 t = \int_1^5 \frac{ds}{v}; \text{ approx.} = \frac{s_5 - s_1}{12}\left[\frac{1}{v_1} + \frac{4}{v_2} + \frac{2}{v_3} + \frac{4}{v_4} + \frac{1}{v_5}\right];$$

while regarding the motion from 5 to N as uniformly retarded (approximately) with initial velocity $= v_5$ and the final $=$ zero, we have (eq. (4), § 56),

$$\Big[_5^N t = 2(s_n - s_5) \div v_5.$$

By adding the three times now found we have the whole time of ascent. In Fig. 67 the dotted curve on the left shows by horizontal ordinates the variation in the velocity as the distance s increases; similarly on the left are ordinates showing the variation of S. The point E, where the velocity is a maximum $= v_m$, may be found by putting $p = 0$, i.e., for $S = A + G$, the accelerating force being $= 0$, see eq. (1). Below E the accelerating force, and consequently the acceleration, is positive; above, negative (i.e., the back-pressure exceeds the steam-pressure). The horizontal ordinates between the line $HEKL$ and the right line RT are proportional to the accelerating force. If by condensation of the steam a vacuum is produced below the piston on its arrival at N, the accelerating force is downward and $= A + G$. [Let the student determine how the detail of this problem would be changed, if the cylinder were horizontal instead of vertical.]

60. Direct Central Impact.—Suppose two masses M_1 and M_2 to be moving in the same right line so that their distance apart continually diminishes, and that when the collision or impact takes place the line of action of the mutual pressure coincides with the line joining their centres of gravity, or centres of

mass, as they may be called in this connection. This is called a direct central impact, and the motion of each mass is variably accelerated and rectilinear during their contact, the only force being the pressure of the other body. The whole mass of each body will be considered concentrated in the centre of mass, on the supposition that all its particles undergo simultaneously the same change of motion in parallel directions. (This is not strictly true; the effect of the pressure being gradually felt, and transmitted in vibrations. These vibrations endure to some extent after the impact.) When the centres of mass cease to approach each other the pressure between the bodies is a maximum and the bodies have a common velocity; after this, if any capacity for restitution of form (elasticity) exists in either body, the pressure still continues, but diminishes in value gradually to zero, when contact ceases and the bodies separate with different velocities. Reckoning the time from the first instant of contact, let t' = duration of the first period, just mentioned; t'' that of the first + the second (restitution). Fig. 68. Let M_1 and M_2 be the masses, and at *any instant during the contact* let v_1 and v_2 be simultaneous values of the velocities of the mass-centres respectively (reckoning velocities positive toward the right),

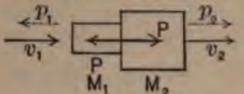


FIG. 68.

and P the pressure (variable). At any instant the acceleration of M_1 is $p_1 = -(P \div M_1)$, while at the same instant that of M_2 is $p_2 = +(P \div M_2)$; M_1 being retarded, M_2 accelerated, in velocity. Hence (eq. II., $p = dv \div dt$) we have

$$M_1 dv_1 = -Pdt; \quad \text{and} \quad M_2 dv_2 = +Pdt. \quad . \quad . \quad (1)$$

Summing all similar terms for the first period of the impact, we have (calling the velocities before impact c_1 and c_2, and the common velocity at instant of maximum pressure C)

$$M_1\int_{c_1}^{C} dv_1 = -\int_0^{t'} Pdt, \text{ i.e., } M_1(C - c_1) = -\int_0^{t'} Pdt; \quad (2)$$

$$M_2\int_{c_2}^{C} dv_2 = +\int_0^{t'} Pdt, \text{ i.e., } M_2(C - c_2) = +\int_0^{t'} Pdt. \quad . \quad (3)$$

The two integrals are identical, numerically, term by term, since the pressure which at any instant accelerates M_2 is numerically equal to that which retards M_1; hence, though we do not know how P varies with the time, we can eliminate the definite integral between (2) and (3) and solve for C. If the impact is *inelastic* (i.e., no power of restitution in either body, either on account of their total inelasticity or damaging effect of the pressure at the surfaces of contact), they continue to move with this common velocity, which is therefore their final velocity. Solving, we have

$$C = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Next, supposing that the impact is *partially elastic*, that the bodies are of the same material, and that the summation $\int_{t'}^{t''} Pdt$ for the second period of the impact bears a ratio, e, to that $\int_0^{t'} Pdt$, already used, a ratio peculiar to the material, if the impact is not too severe, we have, summing equations (1) for the second period (letting V_1 and V_2 = the velocities after impact),

$$M_1 \int_C^{V_1} dv_1 = - \int_{t'}^{t''} Pdt, \text{ i.e., } M_1(V_1 - C) = -e \int_0^{t} Pdt; \quad (5)$$

$$M_2 \int_C^{V_2} dv_2 = + \int_{t'}^{t''} Pdt, \text{ i.e., } M_2(V_2 - C) = +e \int_0^{t} Pdt. \quad (6)$$

e is called the coefficient of restitution.

Having determined the value of $\int_0^{t} Pdt$ from (2) and (3) in terms of the masses and initial velocities, substitute it and that of C, from (4), in (5), and we have (for the final velocities)

$$V_1 = [M_1 c_1 + M_2 c_2 - e M_2 (c_1 - c_2)] \div [M_1 + M_2]; \quad . \quad (7)$$

and similarly

$$V_2 = [M_1 c_1 + M_2 c_2 + e M_1 (c_1 - c_2)] \div [M_1 + M_2]. \quad . \quad (8)$$

For $e = 0$, i.e., *for inelastic impact*, $V_1 = V_2 = C$ in eq. (4); for

$e = 1$, or *elastic impact*, (7) and (8) become somewhat simplified.

To determine e experimentally, let a ball (M_1) of the substance fall upon a very large slab (M_2) of *the same substance*, noting both the height of fall h_1, and the height of rebound H_1. Considering M_2 as $= \infty$, with

$$c = \sqrt{2gh_1}, \quad V_1 = -\sqrt{2gH_1}, \text{ and } c_2 = o,$$

eq. (7) gives

$$-\sqrt{2gH_1} = -e\sqrt{2gh_1}\,; \therefore e = \sqrt{H_1 \div h_1}.$$

Let the student prove the following from equations (2), (3), (5), and (6):

(*a*) For any direct central impact whatever,

$$M_1c_1 + M_2c_2 = M_1V_1 + M_2V_2.$$

[The product of a mass by its velocity being sometimes called its *momentum*, this result may be stated thus:

In any direct central impact the sum of the momenta before impact is equal to that after impact (or at any instant during impact). This principle is called the *Conservation of Momentum*. The present is only a particular case of a more general proposition.

It may be proved C, eq. (4), is the velocity of the centre of gravity of the two masses before impact; the conservation of momentum, then, asserts that this velocity is unchanged by the impact, i.e., by the mutual actions of the two bodies.]

(*b*) The loss of velocity of M_1, and the gain of velocity of M_2, are twice as great when the impact is elastic as when inelastic.

(*c*) If $e = 1$, and $M_1 = M_2$, then $V_1 = -c_2$, and $V_2 = c_1$.

CHAPTER II.

"VIRTUAL VELOCITIES."

61. Definitions.—If a material point is moving in a direction not coincident with that of the resultant force acting (as in curvilinear motion in the next chapter), and any element of its path, ds, projected upon this force; the length of this projection, du, Fig. 69, is called the "VIRTUAL VELOCITY" of the force, since $du \div dt$ may be considered the velocity of the force at this instant, just as $ds \div dt$ is that of the point. The product of the force by its du will be called its *virtual moment*, reckoned $+$ or $-$ according as the direction from O to D is the same as that of the force or opposite.

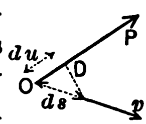


FIG. 69.

62. Prop. I.—*The virtual moment of a force equals the algebraic sum of those of its components.* Fig. 70. Take the direction of ds as an axis X; let P_1 and P_2 be components of P; α_1, α_2, and α their angles with X. Then (§ 16) $P \cos \alpha = P_1 \cos \alpha_1 + P_2 \cos \alpha_2$. Hence $P(ds \cos \alpha) = P_1(ds \cos \alpha_1) + P_2(ds \cos \alpha_2)$. But $ds \cos \alpha$ = the projection of ds upon P, i.e., $= du$; $ds \cos \alpha_1 = du_1$, etc.; $\therefore Pdu = P_1du_1 + P_2du_2$. If in Fig. 70 α_1 were $> 90°$, evidently we would have $Pdu = -P_1du_1 + P_2du_2$, i.e., P_1du_1 would then be negative, and OD_1 would fall behind O; hence the definition of $+$ and $-$ in § 61. For any number of components the proof would be similar, and is equally applicable whether they are in one plane or not.

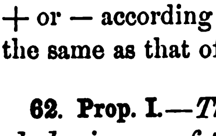


FIG. 70.

63. Prop. II.—*The sum of the virtual moments equals zero, for concurrent forces in equilibrium.*

(If the forces are balanced, the material point is moving in a straight line if moving at all.) The resultant force is zero. Hence, from § 62, $P_1du_1 + P_2du_2 +$ etc. $= 0$, having proper regard to sign, i.e., $\Sigma(Pdu) = 0$.

64. Prop. III.—*The sum of the virtual moments equals zero for any small displacement or motion of a rigid body in equilibrium under non-concurrent forces in a plane; all points of the body moving parallel to this plane.* (Although the kinds of motion of a given rigid body which are consistent with balanced non-concurrent forces have not yet been investigated, we may imagine any slight motion for the sake of the algebraic relations between the different du's and forces.)

FIG. 71.

First, let the motion be a *translation*, all points of the body describing equal parallel lengths $= ds$. Take X parallel to ds; let α_1, etc., be the angles of the forces with X. Then (§ 35) $\Sigma(P \cos \alpha) = 0$; $\therefore ds\Sigma(P \cos \alpha) = 0$; but $ds \cos \alpha_1 = du_1$; $ds \cos \alpha_2 = du_2$; etc.; $\therefore \Sigma(Pdu) = 0$. Q. E. D.

Secondly, let the motion be a *rotation* through a small angle $d\theta$ in the plane of the forces about any point O in that plane, Fig. 72. With O as a pole let ρ_1 be the radius-rector of the point of application of P_1, and a_1 its lever-arm from O; similarly for the other forces. In the rotation each point of application describes a small arc, ds_1, ds_2, etc., proportional to ρ_1, ρ_2, etc., since $ds_1 = \rho_1 d\theta$, $ds_2 = \rho_2 d\theta$, etc. From § 36, $P_1a_1 +$ etc. $= 0$; but from similar triangles $ds_1 : du_1 :: \rho_1 : a_1$; $\therefore a_1 = \rho_1 du_1 \div ds_1 = du_1 \div d\theta$; similarly $a_2 = du_2 \div d\theta$, etc.

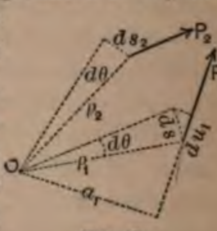


FIG. 72.

Hence we must have $[P_1du_1 + P_2du_2 + \ldots] \div d\theta = 0$, i.e., $\Sigma(Pdu) = 0$. Q. E. D.

Now since any slight displacement or motion of a body may be conceived to be accomplished by a small translation followed by a rotation through a small angle, and since the fore-

going deals only with projections of paths, the proposition is established and is called the *Principle of Virtual Velocities.*

[A similar proof may be used for any slight motion whatever in space when a system of non-concurrent forces is balanced.] Evidently if the path (ds) of a point of application is perpendicular to the force, the virtual velocity (du), and consequently the virtual moment (Pdu) of the force are zero. Hence we may frequently make the displacement of such a character in a problem that one or more of the forces may be excluded from the summation of virtual moments.

65. Connecting-Rod by Virtual Velocities.—Let the effective steam-pressure P be the means, through the connecting-rod and crank (i.e., two links), of raising the weight G *very slowly;* neglect friction and the weight of the links themselves. Consider AB as *free* (see (b) in Fig. 73), BC also, at (c); let the

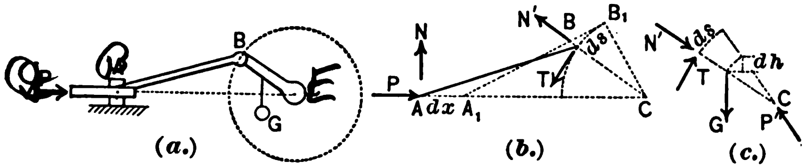


FIG. 73.

"small displacements" of both be *simultaneous* small portions of their ordinary motion in the apparatus. A has moved to A_1 through dx; B to B_1, through ds, a small arc; C has not moved. The forces acting on AB are P (steam-pressure), N (vertical reaction of guide), and N' and T (the tangential and normal components of the crank-pin pressure). Those on BC are N' and T (reversed), the weight G, and the oblique pressure of bearing P'. The motion being slow (or rather the acceleration being small), each of these two systems will be considered as balanced. Now put $\Sigma(Pdu) = 0$ for AB, and we have

$$Pdx + N \times 0 + N' \times 0 - Tds = 0. \quad . \quad . \quad (1)$$

For the simultaneous and corresponding motion of BC, $\Sigma(Pdu) = 0$ gives

$$N' \times 0 + Tds - Gdh + P' \times 0 = 0, \quad . \quad . \quad (2)$$

dh being the vertical projection of G's motion.

From (1) and (2) we have, easily, $Pdx - Gdh = 0$, . (3) which is the same as we might have obtained by putting $\Sigma(Pdu) = 0$ *for the two links together, regarded collectively as a free body*, and describing a small portion of the motion they really have in the mechanism, viz.,

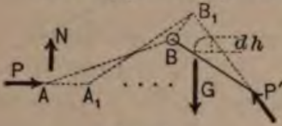


FIG. 74.

$$Pdx + N \times 0 - Gdh + P' \times 0 = 0. \quad . \quad . \quad (4)$$

We may therefore announce the—

66. Generality of the Principle of Virtual Velocities.—*If any mechanism of flexible inextensible cords, or of rigid bodies jointed together, or both, at rest, or in motion with very small accelerations, be considered free collectively (or any portion of it), and all the external forces put in; then (disregarding mutual frictions) for a small portion of its prescribed motion,* $\Sigma(Pdu)$ *must* $= 0$, *in which the du, or virtual velocity, of each force, P is the projection of the path of the point of application upon the force* (the product, Pdu, being + or — according to § 61).

67. *Example.*—In the problem of § 65, having given the weight G, required the proper steam-pressure (effective) P to hold G in equilibrium, or to raise it uniformly, if already in motion, for a *given position of the links.* That is, Fig. 75, given a, r, c, α, and β, required the ratio $dh : dx$; for, from equation (3), § 65, $P = G(dh : dx)$. The projections of dx and ds upon AB will be equal, since $AB = A_1B_1$, and makes an (infinitely) small angle with A_1B_1, i.e., $dx \cos \alpha = ds \cos (\beta - \alpha)$. Also, $dh = (c : r)ds \sin \beta$.

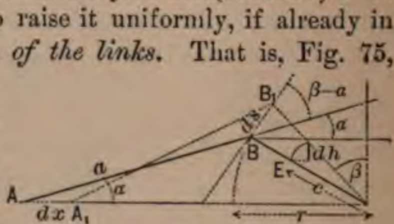


FIG. 75.

Eliminating ds, we have,

$$\frac{dh}{dx} = \frac{c \sin \beta \cos \alpha}{r \cos (\beta - \alpha)}; \quad \therefore P = G\frac{c \sin \beta \cos \alpha}{r \cos (\beta - \alpha)}.$$

68. When the acceleration of the parts of the mechanism is not practically zero, $\Sigma(Pdu)$ will not $= 0$, but a function of the masses and velocities to be explained in the chapter on Work, Energy, and Power. If friction occurs at moving joints, enough "free bodies" should be considered that no free body extend beyond such a joint; it will be found that this friction cannot be eliminated in the way in which T and N' were, in § 65.

69. Additional Problems; to be solved by "virtual velocities."
PROBLEM 1.—Find relations between the forces acting on a straight lever in equilibrium; also, on a bent lever.

PROBLEM 2.—When an ordinary copying-press is in equilibrium, find the relation between the force applied horizontally and tangentially at the circumference of the wheel, and the vertical resistance under the screw-shaft.

CHAPTER III.

CURVILINEAR MOTION OF A MATERIAL POINT.

[Motion in a plane, only, will be considered in this chapter.]

70. Parallelogram of Motions.—It is convenient to regard the curvilinear motion of a point in a plane as compounded, or made up, of two independent rectilinear motions parallel respectively to two co-ordinate axes X and Y, as may be explained thus: Fig. 76. Consider the drawing-board CD as fixed, and let the head of a T-square move from A toward B along the edge according to any law whatever, while a pencil moves from M toward Q along the blade. The result is a curved line on the board, whose form depends on the character of the two X and Y *component motions*, as they may be called. If in a time t_1 the T-square head has moved an X distance $= MN$, and the pencil simultaneously a Y distance $= MP$, by completing the parallelogram on these lines, we obtain R, the position of the point on the board at the end of the time t_1. Similarly, at the end of the time t_1' we find the point at R'.

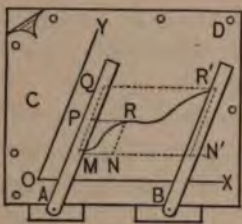


FIG. 76.

71. Parallelogram of Velocities.—Let the X and Y motions be *uniform*, required the resulting motion. Fig. 77. Let c_x and c_y be the constant uniform X and Y velocities. Then in any time, t, we have $x = c_x t$ and $y = c_y t$; whence we have, eliminating t, $x \div y = c_x \div c_y =$ constant, i.e., x is proportional to y, i.e., the path is a straight line. Laying off $OA = c_x$, and $AB = c_y$, B is a point of the path, and OB is the distance described by the point in the first

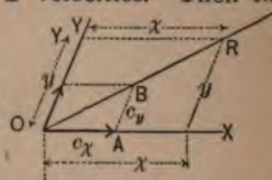


FIG. 77.

second. Since by similar triangles $\overline{OR} : x :: \overline{OB} : c_x$, we have also $\overline{OR} = \overline{OB} \cdot t$; hence the resultant motion is uniform, and its velocity, $\overline{OB} = c$, *is the diagonal of the parallelogram formed on the two component velocities.*

Corollary.—If the resultant motion is curved, the direction and velocity of the motion at any point will be given by the diagonal formed on the component velocities at that instant. The direction of motion is, of course, a tangent to the curve.

72. Uniformly Accelerated X and Y Motions.—The *initial velocities of both being zero.* Required the resultant motion. Fig. 78. From § 56, eq. (2) (both c_x and c_y being $= 0$), we have $x = \frac{1}{2}p_x t^2$ and $y = \frac{1}{2}p_y t^2$, whence $x \div y = p_x \div p_y =$ constant, and the resultant motion is in a straight line. Conceive lines laid off from O on X and Y to represent p_x and p_y to scale, and form a parallelogram on them. From similar triangles (OR being the distance described in the resultant motion in any time t), $\overline{OR} : x :: \overline{OB} : p_x$; $\therefore$ $\overline{OR} = \frac{1}{2}\overline{OB}t^2$. Hence, from the form of this last equation, the resultant motion is uniformly accelerated, and its acceleration is $\overline{OB} = p$, (on the same scale as p_x and p_y).

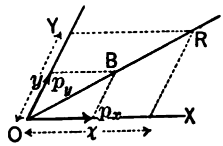


FIG. 78.

This might be called the parallelogram of accelerations, but is really a parallelogram of forces, if we consider that a free material point, initially at rest at O, and acted on simultaneously by constant forces P_x and P_y (so that $p_x = P_x \div M$ and $p_y = P_y \div M$), must begin a uniformly accelerated rectilinear motion in the direction of the resultant force, having no initial velocity in any direction.

73. In general, considering the point hitherto spoken of as a *free material point*, under the action of one or more forces, in view of the foregoing, and of Newton's second law, given the initial velocity in amount and direction, the starting-point, the initial amounts and directions of the acting forces and the

laws of their variation if they are not constant, we can resolve them into a single X and a single Y force at any instant, determine the X and Y motions independently, and afterwards the resultant motion. The resultant force is never in the direction of the tangent to the path (except at a point of inflection). The relations which its amount and direction at any instant bear to the velocity, the rate of change of that velocity, and the radius of curvature of the path will appear in the next paragraph.

74. General Equations for the curvilinear motion of a material point in a plane.—The motion will be considered resulting from the composition of independent X and Y motions, X and Y being perpendicular to each other. Fig. 79. In two consecutive **equal** times (each $= dt$), let dx and $dx' =$ small spaces due to the X motion; and dy and $CK = dy'$, due to the Y motion. Then ds and ds' are two consecutive elements of the curvilinear motion. Prolong ds, making $BE = ds$; then $EF = d^2x$, $CF = d^2y$, and $CO = d^2s$ (EO being perpendicular to BE). Also draw CL perpendicular to BG and call CL d^2n. Call the velocity of the X motion v_x, its acceleration p_x; those of the Y motion, v_y and p_y. Then,

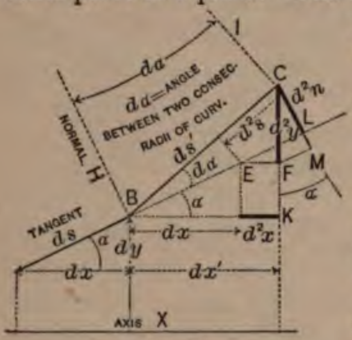


Fig. 79.

$$v_x = \frac{dx}{dt}; \quad v_y = \frac{dy}{dt}; \quad p_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}; \quad \text{and } p_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}.$$

For the velocity along the curve (i.e., tangent) $v = ds \div dt$, we shall have, since $ds^2 = dx^2 + dy^2$,

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = v_x^2 + v_y^2. \quad . \quad . \quad (1)$$

Hence v is the diagonal formed on v_x and v_y (as in § 71). Let $p_t =$ the acceleration of v, i.e., the *tangential acceleration*,

then $p_t = d^2s \div dt^2$, and, since $d^2s =$ the sum of the projections of EF and CF on BC, i.e., $d^2s = d^2x \cos\alpha + d^2y \sin\alpha$, we have

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\cos\alpha + \frac{d^2y}{dt^2}\sin\alpha;\ \text{i.e., } p_t = p_x\cos\alpha + p_y\sin\alpha. \quad (2)$$

By **Normal Acceleration** we mean the rate of change of the velocity in the direction of the normal. In describing the element $AB = ds$, no progress has been made in the direction of the normal BH i.e., there is *no velocity* in the direction of the normal; but in describing BC (on account of the new direction of path) the point has progressed a distance CL (call it d^2n) in the direction of the old normal BH (though none in that of the new normal CI). Hence, just as the tang. acc. $= \dfrac{ds' - ds}{dt^2} = \dfrac{d^2s}{dt^2}$, so the normal accel. $= \dfrac{CL - \text{zero}}{dt^2} = \dfrac{d^2n}{dt^2}$.
It now remains to express this normal acceleration ($= p_n$) in terms of the X and Y accelerations. From the figure, $\overline{CL} = \overline{CM} - ML$, i.e.,

$$d^2n = d^2y\cos\alpha - d^2x\sin\alpha\ \{\text{since } EF = d^2x\};$$

$$\therefore \frac{d^2n}{dt^2} = \frac{d^2y}{dt^2}\cos\alpha - \frac{d^2x}{dt^2}\sin\alpha.$$

Hence $\qquad p_n = p_y\cos\alpha - p_x\sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad (3)$

The norm. acc. may also be expressed in terms of the tang. velocity v, and the radius of curvature r, as follows:
$ds' = r\,d\alpha$, or $d\alpha = ds' \div r$; also $d^2n = ds'd\alpha, \doteq ds'^2 \div r$,

$$\text{i.e., } \frac{d^2n}{dt^2} = \left(\frac{ds'}{dt}\right)^2\frac{1}{r}, \quad \text{or} \quad p_n = \frac{v^2}{r}. \quad . \quad . \quad . \quad . \quad (4)$$

If now, Fig. 80, we resolve the forces $X = Mp_x$ and $Y = Mp_y$, which at this instant account for the X and Y accelerations ($M =$ mass of the material point), into components along the tangent and normal to the curved path, we shall have, as *their equivalent*, a tangential force

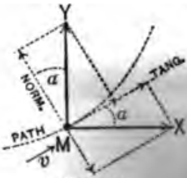


FIG. 80.

$$T = Mp_x\cos\alpha + Mp_y\sin\alpha,$$

and a normal force

$$N = Mp_y \cos \alpha - Mp_x \sin \alpha.$$

But [see equations (2), (3), and (4)] we may also write

$$T = Mp_t = M\frac{dv}{dt}; \quad \text{and} \quad N = Mp_n = M\frac{v^2}{r}. \quad . \quad (5)$$

Hence, if a free material point is moving in a curved path, the sum of the tangential components of the acting forces must equal (the mass) $\times$ tang. accel.; *that of the normal components,* = (the mass) $\times$ normal accel. = (mass) $\times$ (square of veloc. in path) $\div$ (rad. curv.).

It is evident, therefore, that the resultant force (= diagonal on T and N or on X and Y, Fig. 80) *does not act along the tangent* at any point, but toward the concave side of the path; unless $r = \infty$.

Radius of curvature.—From the line above eq. (4) we have $d^2n = ds^2 \div r$; hence (line above eq. (3)), $ds^2 \div r = d^2y \cos \alpha - d^2x \sin \alpha$; but $\cos \alpha = dx \div ds$, and $\sin \alpha = dy \div ds$,

$$\therefore \frac{ds^2}{r} = d^2y\frac{dx}{ds} - d^2x\frac{dy}{ds}; \quad \text{or} \quad \frac{ds^2}{r} = dx^2\left[\frac{dxd^2y - dyd^2x}{dx^3}\right];$$

$$\text{i.e.,} \frac{ds^2}{r} = dx^2 d\left[\frac{dy}{dx}\right] = dx^2 d(\tan \alpha),$$

$$\therefore r = \left(\frac{ds}{dt}\right)^3 \div \left[\left(\frac{dx}{dt}\right)^2 \frac{d \tan \alpha}{dt}\right];$$

or,

$$r = v^3 \div \left[v_x^2 \frac{d \tan \alpha}{dt}\right]. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which is equally true if, for v_x and $\tan \alpha$, we put v_y and $\tan (90° - \ )$, respectively.

Change in the velocity square.—Since the tangential acceleration $\frac{dv}{dt} = p_t$, we have $ds\frac{dv}{dt} = p_t ds$; i.e.,

$$\frac{ds}{dt}dv = p_t ds, \quad \text{or} \quad vdv = p_t ds \quad \text{and} \quad \therefore \frac{v^2 - c^2}{2} = \int p_t ds. \quad (7)$$

having integrated between any initial point of the curve where $v = c$, and any other point where $v = v$. This is nothing more than equation (III.), of § 50.

75. Normal Acceleration. Second Method.—Fig. 81. Let C be the centre of curvature and $OD = 2r$. Let OB' be a portion of the osculatory parabola (vertex at O; any osculatory curve will serve). When ds is described, the distance passed over in the direction of the normal is AB; for $2ds$, it would be $A'B' = 4AB$ (i.e., as the square of OB'; property of a parabola), and so on. Hence the motion along the normal is uniformly accelerated with initial velocity $= 0$, since the distance AB, varies as the *square of the time* (considering the motion along the curve of uniform velocity, so that the distance OB is directly as the time). If p_n denote the accel. of this uniformly accelerated motion, its initial velocity being $= 0$, we have (eq. 2, § 56) $\overline{AB} = \frac{1}{2}p_n dt^2$, i.e., $p_n = 2\overline{AB} \div dt^2$. But from the similar triangles ODB and OAB we have, $\overline{AB} : ds :: ds : 2r$, hence $2\overline{AB} = ds^2 \div r$, $\therefore p_n = ds^2 \div rdt^2 = v^2 \div r$.

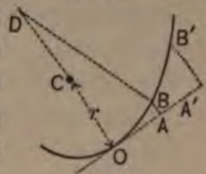


FIG. 81.

76. Uniform Circular Motion. Centripetal Force.—The velocity being constant, p_t must be $= 0$, and $\therefore T$ (or ΣT if there are several forces) must $= 0$. The resultant of all the forces, therefore, must be a normal force $= (Mc^2 \div r) =$ a constant (eq. 5, § 74). This is called the "deviating force," or "centripetal force;" without it the body would continue in a straight line. Since forces always occur in pairs (§ 3), a "centrifugal force," equal and opposite to the "centripetal" (one being the reaction of the other), will be found among the forces acting on the body to whose constraint the deviation of the first body from its natural straight course is due. For example, the attraction of the earth on the moon acts as a centripetal or deviating force on the latter, while the equal and opposite force *acting on the earth* may be called the centrifugal. If a small block moving on a smooth horizontal table is gradually turned from its straight course AB by a fixed circular guide, tangent to AB at B, the pressure of the guide against the block is the centripetal force $Mc^2 \div r$ [illegible] *toward* the centre of curvature, while

the centrifugal force $Mc^2 \div r$ is the pressure of the block against the guide, directed *away* from that centre. The centrifugal force, then, is never found among the forces acting on the body whose circular motion we are dealing with.

The Conical Pendulum, or governor-ball.—Fig. 82. If a material point of mass $= M = G \div g$, suspended on a cord of

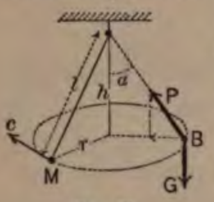


FIG. 83.

length $= l$, is to maintain a uniform circular motion in a horizontal plane, with a given radius r, under the action of gravity and the cord, required the velocity c to be given it. At B we have the body free. The only forces acting are G and the cord-tension P. The sum of their normal components, i.e., ΣN, must $= Mc^2 \div r$, i.e., $P \sin \alpha = Mc^2 \div r$; but, since Σ (vert. comps.) $= 0$, $P \cos \alpha = G$. Hence $G \tan \alpha = Gc^2 \div gr$; $\therefore c = \sqrt{gr \tan \alpha}$. Let $u =$ number of revolutions per unit of time, then $u = c \div 2\pi r = \sqrt{g} \div 2\pi \sqrt{h}$; i.e., is inversely proportional to the (vertical projection)$^{\frac{1}{2}}$ of the cord-length. The time of one revolution is $= 1 \div u$.

Elevation of the outer rail on railroad curves (considerations of traction disregarded).—Consider a single car as a material point, and *free*, having a given velocity $= c$. P is the rail-pressure against the wheels. So long as the car follows the track the resultant R of P and G must point toward the centre of curvature and have a value $= Mc^2 \div r$. But $R = G \tan \alpha$, whence $\tan \alpha = c^2 \div gr$. If therefore the ties are placed at this angle α with the horizontal, the pressure

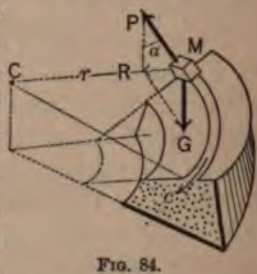


FIG. 84.

will come upon the tread and not on the flanges of the wheels; in other words, the car will not leave the track. (This is really the same problem as the preceding.)

Apparent weight of a body at the equator.—This is less than the true weight or attraction of the earth, on account of the uniform circular motion of the body with the earth in its diurnal rotation. If the body hangs from a spring-balance,

whose indication is G lbs. (apparent weight), while the true attraction is G' lbs., we have $G' - G = Mc^2 \div r$. For M we may use $G \div g$ (apparent values); for r about 20,000,000 ft.; for c, 25,000 miles in 24 hrs., reduced to feet per second. It results from this that G is $< G'$ by $\frac{1}{289}G'$ nearly, and (since $17^2 = 289$) hence if the earth revolved on its axis seventeen times as fast as at present, G would $= 0$, i.e., bodies would apparently have no weight, the earth's attraction on them being just equal to the necessary centripetal or deviating force necessary to keep the body in its orbit.

Centripetal force at any latitude.—If the earth were a homogeneous liquid, and at rest, its form would be spherical; but when revolving uniformly about the polar diameter, its form of relative equilibrium (i.e., no motion of the particles relatively to each other) is nearly ellipsoidal, the polar diameter being an axis of symmetry.

Lines of attraction on bodies at its surface do not intersect in a common point, and the centripetal force requisite to keep a suspended body in its orbit (a small circle of the ellipsoid), at any latitude β is the resultant of the attraction or true weight G' directed (nearly) toward the centre, and of G the tension of the string. Fig. 85. G is the apparent weight, indicated by a spring-balance and MA is its line of action (plumb-line) normal to the ocean surface. Evidently the apparent weight, and consequently g, are less than the true values, since N must be perpendicular to the polar axis, while the true values themselves, varying inversely as the square of MC, *decrease* toward the equator, hence the apparent values decrease still more rapidly as the latitude diminishes. The following equation gives the apparent g for any latitude ... very nearly (units foot and second):

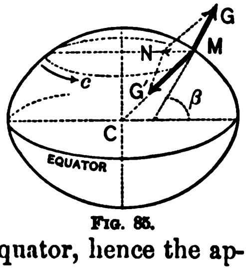


FIG. 85.

$$g = 32.18 \ldots \cdot 821 \cos 2\beta.$$

... h for practical purposes.)

... rest, but moving about

the sun, and also about the centre of gravity of the moon and earth, the form of the ocean surface is periodically varied, i.e., the phenomena of the tides are produced.

77. Cycloidal Pendulum.—This consists of a material point at the extremity of an imponderable, flexible, and inextensible cord of length $= l$, confined to the arc of a cycloid in a vertical plane by the cycloidal evolutes shown in Fig. 86. Let the oscillation begin (from rest) at A, a height $= h$ above O

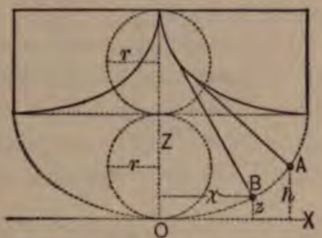


FIG. 86.

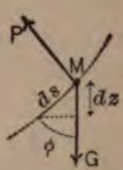


FIG. 87.

the vertex. On reaching any lower point, as B (height $= z$ above O), the point has acquired some velocity v, which is at this instant increasing at some rate $= p_t$. Now consider the point free, Fig. 87; the forces acting are P the cord-tension, normal to path, and G the weight, at an angle φ with the path. From § 74, eq. (5), $\Sigma T = Mp_t$ gives

$$G \cos \varphi + P \cos 90° = (G \div g)p_t; \therefore p_t = g \cos \varphi$$

Hence (eq. (7), § 74), $vdv = p_t ds$ gives

$vdv = g \cos \varphi ds$; but $ds \cos \varphi = -dz$; $\therefore vdv = -gdz$.

Summing between A and B, we have

$$\Big[_0^v \tfrac{1}{2}v^2 = -g\int_h^z dz; \quad \text{or} \quad v^2 = 2g(h-z);$$

the same as if it had fallen freely from rest through the height $h - z$. (*This result evidently applies to any form of path when, besides the weight G, there is but one other force, and that always normal to the path.*)

From $\Sigma N = Mv^2 \div r$, we have $P - G \sin \varphi = Mv^2 \div r$,

whence P, the cord-tension at any point, may be found (here $r =$ the radius of curvature at any point $=$ length of straight portion of the cord).

To find the time of passing from A to O, a half-oscillation, substitute the above value of v^2 in $v = ds \div dt$, putting $ds^2 = dx^2 + dz^2$, and we have $dt^2 = (dx^2 + dz^2) \div [2g(h - z)]$. To find dx in terms of dz, differentiate the equation of the curve, which in this position is

$$x = r \text{ ver. sin.}^{-1} (z \div r) + \sqrt{2rz - z^2};$$

whence

$$dx = \frac{rdz}{\sqrt{2rz - z^2}} + \frac{(r - z)dz}{\sqrt{2rz - z^2}} = \frac{(2r - z)dz}{\sqrt{2rz - z^2}};$$

$$\therefore dx^2 = \left[\frac{2r}{z} - 1\right]dz^2$$

($r =$ radius of the generating circle). Substituting, we have

$$dt = \sqrt{\frac{r}{g}} \cdot \frac{(-dz)}{\sqrt{hz - z^2}};$$

$$\therefore \Big[_A^O t = \sqrt{\frac{r}{g}} \int_0^h \frac{dz}{\sqrt{hz - z^2}} = \sqrt{\frac{r}{g}} \Big[_0^h \text{ver. sin.}^{-1} \frac{z}{\frac{1}{2}h} = \pi\sqrt{\frac{r}{g}}.$$

Hence the whole oscillation occupies a time $= \pi \sqrt{l \div g}$ (since $l = 4r$). This is independent of h, i.e., the oscillations are *isochronal*. This might have been proved by showing that p is proportional to OB *measured along the curve;* i.e., that the motion is *harmonic*. (§ 59, Prob. 2.)

78. Simple Circular Pendulum.—If the material point oscillates in the arc of a circle, Fig. 88, proceeding as in the preceding problem, we have finally, after integration by series, as the time of a full oscillation,

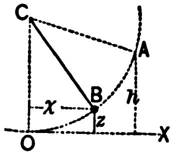


FIG. 88.

$$2\Big[_A^O t = \pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{8} \cdot \frac{h}{l} + \frac{9}{256}\frac{h^2}{l^2} + \dots\right].$$

Hence for a small h the time is nearly $\pi \sqrt{l \div g}$, and the os-

cillations nearly isochronal. (For the Compound Pendulum, see § 117.)

79. Change in the Velocity Square.—From eq. (7), § 74, we have $\frac{1}{2}(v^2 - c^2) = \int p_t ds$. But, from similar triangles, du being the projection of any ds of the path upon the resultant force R at that instant, $Rdu = Tds$ (or, Prin. of Virt. Vels. § 62, $Rdu = Tds + N \times 0$). T and N are the tangential and normal components of R. Fig. 89. Hence, finally,

$$\tfrac{1}{2}Mv^2 - \tfrac{1}{2}Mc^2 = \int Rdu, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

for all elements of the curve between any two points. In general R is different in amount and direction for each ds of the path, but du is the distance through which R acts, **in its own direction**, while the body describes any ds; Rdu is called the **work done** by R when ds is described by the body. The above equation is read: *The difference between the initial and final kinetic energy of a body = the work done by the resultant force in that portion of the path.*

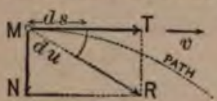


FIG. 89.

(These phrases will be further spoken of in Chap. VI.)

Application of equation (*a*) *to a planet in its orbit about the sun.*—Fig. 90. Here the only force at any instant is the attraction of the sun $R = C \div u^2$ (see Prob. 3, § 59), where C is a constant and u the variable radius vector. As u diminishes, v increases, therefore dv and du have contrary signs; hence equation (*a*) gives (c being the velocity at some initial point O)

FIG. 90.

$$\frac{1}{2}Mv_1^2 - \frac{1}{2}Mc^2 = -C\int_{u_0}^{u_1}\frac{du}{u^2} = C\left[\frac{1}{u_1} - \frac{1}{u_0}\right]; \; (b)$$

$$\therefore v_1 = \sqrt{c^2 + \frac{2C}{M}\left[\frac{1}{u_1} - \frac{1}{u_0}\right]},$$ which is independent of the direction of the initial velocity c.

NOTE.—If u_0 were = infinity, the last member of equation (b) whuld reduce to $C \div u_1$, and is numerically the quantity called **potential** in the theory of electricity.

Application of eq. (*a*) *to a projectile in vacuo.*—G, the body's weight, is the only force acting, and therefore $= R$, while $M = G \div g$. Therefore equation (*a*) gives

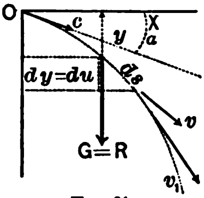


FIG. 91.

$$\frac{G}{g} \cdot \frac{v_1^2 - c^2}{2} = G\int_0^{y_1} dy = Gy;$$

$\therefore v_1 = \sqrt{c^2 + 2gy_1}$, which is independent of the angle, α, of projection.

Application of equation (*a*) to a body sliding, *without friction*, on a fixed curved guide in a vertical plane; initial velocity $= c$ at O.—Since there is some pressure at each point between the body and the guide, to consider the body *free* in space, we must consider the guide removed and that the body describes the given curve as a result of the action of the two forces, its weight G, and the pressure P, of the guide against the body. G is constant, while P varies from point to point, though always (since there is no friction) *normal to curve.*

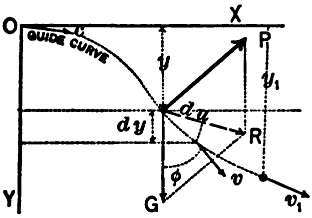


FIG 92.

At any point, R being the resultant of G and P, project ds upon R, thus obtaining du; on G, thus obtaining dy; on P, thus obtaining zero. But by the *principle of virtual velocities* (see § 62) we have $Rdu = Gdy + P \times \text{zero} = Gdy$, which substituted in eq. (*a*) gives

$$\frac{G}{g}\frac{1}{2}(v_1^2 - c^2) = \int_0^{y_1} Gdy = G\int_0^{y_1} dy = Gy_1; \quad \therefore v_1 = \sqrt{c^2 + 2gy_1}$$

and therefore depends only on the *vertical distance* fallen through and the initial velocity, i.e., is *independent of the form of the guide.*

As to the value of P, the mutual pressure between the guide and body at any point, since ΣN must equal $Mv^2 \div r$, r being the variable radius of curvature, we have, as in § 77,

$$P - G \sin \varphi = Mv^2 \div r; \quad \therefore P = G[\sin \varphi + (v^2 \div gr)].$$

As, in general, φ and r are different from point to point of

the path, P is not constant. (The student will explain how P may be negative on parts of the curve, and the meaning of this circumstance.)

80. Projectiles in Vacuo.—A ball is projected into the air (whose resistance is neglected, hence the phrase *in vacuo*) at an angle $= \alpha_0$ with the horizontal; required its path; assuming it confined to a vertical plane. Resolve the motion into independent horizontal (X) and vertical (Y) motions, G, the weight, the only force acting, being correspondingly replaced by its horizontal component $=$ zero, and its vertical component $= -G$. Similarly the initial velocity along $X = c_x = c \cos \alpha_0$, along Y, $= c_y = c \sin \alpha_0$. The X acceleration $= p_x = 0 \div M = 0$, i.e., the X motion is uniform, the velocity v_x remains $= c_x = c \cos \alpha_0$ at all points, hence, reckoning the time from O, at the end of any time t we have

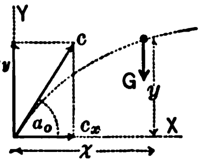


Fig. 93.

$$x = c(\cos \alpha)t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the Y motion, $p_y = (-G) \div M = -g$, i.e., it is uniformly retarded, the initial velocity being $c_y = c \sin \alpha_0$; hence, after any time t, the Y velocity will be (see § 56) $v_y = c \sin \alpha_0 - gt$, while the distance

$$y = c(\sin \alpha_0)t - \tfrac{1}{2}gt^2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Between (1) and (2) we may eliminate t, and obtain as the equation of the trajectory or path

$$y = x \tan \alpha_0 - \frac{gx^2}{2c^2 \cos^2 \alpha_0}.$$

For brevity put $c^2 = 2gh$, h being the ideal *height due to the velocity* c, i.e., $c^2 \div 2g$ (see § 53; if the ball were directed vertically upward, a height $h = c^2 \div 2g$ would be actually attained, α being $= 90°$), and we have

$$y = x \tan \alpha_0 - \frac{x^2}{4h \cos^2 \alpha_0}. \quad . \quad . \quad . \quad . \quad (3)$$

This is easily shown to be the equation of a parabola, with its axis vertical.

The horizontal range.—Fig. 94. Putting $y = 0$ in equation (3), we obtain

$$x\left[\tan \alpha_0 - \frac{x}{4h \cos^2 \alpha_0}\right] = 0,$$

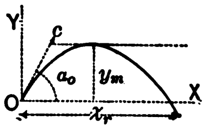


FIG. 94.

which is satisfied both by $x = 0$ (i.e., at the origin), and by $x = 4h \cos \alpha_0 \sin \alpha_0$. Hence the horizontal range for a given c and α_0 is $x_r = 4h \cos \alpha_0 \sin \alpha_0 = 2h \sin 2\alpha_0$.

For $\alpha_0 = 45°$ this is a maximum (c remaining the same), being then $= 2h$. Also, since $\sin 2\alpha_0 = \sin (180° - 2\alpha_0) = \sin 2(90° - \alpha_0)$, therefore any two complementary angles of projection give the same horizontal range.

Greatest height of ascent; that is, the value of y maximum, $= y_m$.—Fig. 94. Differentiate (3), obtaining

$$\frac{dy}{dx} = \tan \alpha_0 - \frac{x}{2h \cos^2 \alpha_0},$$

which, put $= 0$, gives $x = 2h \sin \alpha_0 \cos \alpha_0$, and this value of x in (3) gives $y_m = h \sin^2 \alpha_0$.

(Let the student obtain this more simply by considering the Y motion separately.)

To strike a given point; c being given and α_0 required.—Let x' and y' be the co-ordinates of the given point, and α_0' the unknown angle of projection. Substitute these in equation (3), h being known $= c^2 \div 2g$, and we have

$$y' = x' \tan \alpha'_0 - \frac{x'^2}{4h \cos^2 \alpha_0'}. \quad \text{Put } \cos^2 \alpha_0' = \frac{1}{1 + \tan^2 \alpha_0'},$$

and solve for $\tan \alpha_0'$, whence

$$\tan \alpha'_0 = \left[2h \pm \sqrt{4h^2 - x'^2 - 4hy'}\right] \div x'. \quad . \quad . \quad (4)$$

Evidently, if the quantity under the radical in (4) is negative, $\tan \alpha_0'$ is imaginary, i.e., the given point is *out of range* with the given velocity of projection $c = \sqrt{2gh}$; if positive, $\tan \alpha_0'$ has two values, i.e., two trajectories may be passed through the point; while if it is zero, $\tan \alpha_0'$ has but one value.

The envelope, for all possible trajectories having the same

initial velocity c (and hence the same h); i.e., the curve tangent to them all, has but one point of contact with any one of them; hence each point of the envelope, Fig. 95, must have co-ordinates satisfying the condition, $4h^2 - x'^2 - 4hy' = 0$; i.e. (see equation (4)), that there is but one trajectory belonging to it. Hence, dropping primes, the equation of the envelope is $4h^2 - x^2 - 4hy = 0$. Now take O'' as a new origin, a new horizontal axis X'', and reckon y'' positive downwards; i.e., substitute $x = x''$ and $y = h - y''$. The equation now becomes $x''^2 = 4hy''$; evidently the equation of a parabola whose axis is vertical, whose vertex is at O'', and whose parameter $= 4h =$ double the maximum horizontal range. O is therefore its focus.

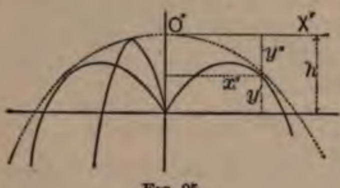


FIG. 95.

The range on an inclined plane.—Fig. 96. Let OC be the trace of the inclined plane; its equation is $y = x \tan \beta$, which, combined with the equation of the trajectory (eq. 3), will give the co-ordinates of their intersection C. That is, substitute $y = x \tan \beta$ in (3) and solve for x, which will be the abscissa x_1, of C. This gives

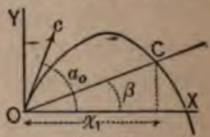


FIG. 96.

$$\frac{x_1}{4h \cos^2 \alpha} = \tan \alpha_0 - \tan \beta = \frac{\sin \alpha_0}{\cos \alpha_0} - \frac{\sin \beta}{\cos \beta} = \frac{\sin (\alpha_0 - \beta)}{\cos \alpha_0 \cos \beta};$$

$\therefore x_1 = 4h \cos \alpha_0 \sin (\alpha_0 - \beta) \div \cos \beta$, and the range $\overline{OC}$, which $= x_1 \div \cos \beta$, is $= (4h \div \cos^2 \beta) \cos \alpha_0 \sin (\alpha_0 - \beta)$. (5)

The maximum range on a given inclined plane, β, c (and $\therefore h$), remaining constant, while α_0 varies.—That is, required the value of α_0 which renders $\overline{OC}$ a maximum. Differentiating (5) with respect to α_0, putting this derivative $= 0$, we have $[4h \div \cos^2 \beta] [\cos \alpha_0 \cos (\alpha_0 - \beta) - \sin \alpha_0 \sin (\alpha_0 - \beta)] = 0$; whence $\cos [\alpha_0 + (\alpha_0 - \beta)] = 0$; i.e., $2\alpha_0 - \beta = 90°$; or, $\alpha_0 = 45° + \frac{1}{2}\beta$, for a maximum range. By substitution this maximum becomes known.

The velocity at any point of the path is $v = \sqrt{v_x^2 + v_y^2} =$

$\sqrt{c^2 - 2ctg \sin \alpha_0 + g^2t^2}$ (see the first part of this § 80); while the *time of passage* from O to any point whose abscissa is x is $t = x \div c \cos \alpha_0$; obtained from equation (1). E.g., to reach the point B, Fig. 94, we put $x = x_r = 4h \sin \alpha \cos \alpha$, and obtain $t_r = 2c \sin \alpha_0 \div g$. This will give the velocity at $B = \sqrt{c^2} = c$.

81. Actual Path of Projectiles.—Small jets of water, so long as they remain unbroken, give close approximations to parabolic paths, as also any small dense object, e.g., a ball of metal, having a moderate initial velocity. The course of a cannon-ball, however, with a velocity of 1200 to 1400 feet per second is much affected by the resistance of the air, the descending branch of the curve being much steeper than the ascending; see Fig. 96*a*. The equation of this curve has not yet been determined, but only the expression for the slope (i.e., $dy : dx$) at any point. See Professor Bartlett's Mechanics, § 151 (in which the body is a sphere having no motion of rotation). Swift rotation about an axis, as well as an unsymmetrical form with reference to the direction of motion, alters the trajectory still further, and may deviate it from a vertical plane. The presence of wind would occasion increased irregularity. See Johnson's Encyclopædia, article "Gunnery."

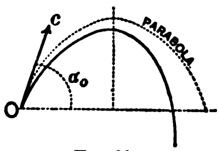
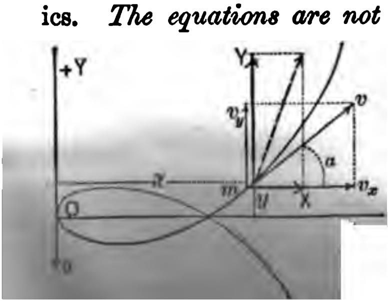


FIG. 96*a*.

82. Special Problem (imaginary; from Weisbach's Mechanics. *The equations are not homogeneous*).—Suppose a material point, mass $= M$, to start from the point O, Fig. 97, with a velocity $= 9$ feet per second along the $-Y$ axis, being subjected thereafter to a constant attractive X force, of a value X $12M$, and to a variable Y increasing with the time ds, reckoned from O),



viz., $Y = 8Mt$. Required the path, etc. For the X motion we have $p_x = X \div M = 12$, and hence

$$\int^{v_x} dv_x = \int_0^t p_x dt = 12\int_0^t dt\text{; i.e., } v_x = 12t;$$

$$\text{and } \int_0^x dx = \int_0^t v_x dt\text{; i.e., } x = 12\int_0^t tdt = 6t^2. \quad . \quad (1)$$

For the Y motion $p_y = Y \div M = 8t$, $\therefore \int_{-9}^{v_y} dv_y = 8\int_0^t tdt$;

$$\text{i.e., } v_y + 9 = 4t^2, \quad \text{and} \quad \int_0^y dy = \int_0^t v_y dt;$$

$$y = 4\int_0^t t^2dt - 9\int_0^t dt, \quad \text{or} \quad y = \tfrac{4}{3}t^3 - 9t. \quad . \quad . \quad (2)$$

Eliminate t between (1) and (2), and we have, as the *equation of the path*,

$$y = \pm\frac{4}{3}\left(\frac{x}{6}\right)^{\frac{3}{2}} \mp 9\left(\frac{x}{6}\right)^{\frac{1}{2}}; \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which indicates a curve of the third order.

The velocity at any point is (see § 74, eq. (1))

$$v = \sqrt{v_x^2 + v_y^2} = 4t^2 + 9. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The length of curve measured from O will be (since $v = ds \div dt$)

$$s = \int_0^s ds = \int_0^t vdt = 4\int_0^t t^2dt + 9\int_0^t dt = \tfrac{4}{3}t^3 + 9t. \quad (5)$$

The *slope*, $\tan\alpha$, at any point $= v_y \div v_x = (4t^2 - 9) \div 12t$,

$$\text{and } \therefore \frac{d \tan\alpha}{dt} = \frac{4t^2 + 9}{12t^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The radius of curvature at any point (§ 74, eq. (6)), substituting $v_x = 12t$, also from (4) and (6), is

$$r = v^3 \div \left[v_x^2\frac{d\tan\alpha}{dt}\right] = \frac{1}{12}[4t^2 + 9]^2, \quad . \quad . \quad (7)$$

and the *normal acceleration* $= v^2 \div r$ (eq. (4), § 74), becomes from (4) and (7) $p_n = 12$ (ft. per square second), a *constant*. Hence the centripetal or deviating force at any point, i.e., the

ΣN of the forces X and Y, is *the same at all points*, and $= Mv^2 \div r = 12M$.

From equation (3) it is evident that the curve is symmetrical about the axis X. Negative values of t and s would apply to points on the dotted portion in Fig. 97, since the body may be considered as having started at any point whatever, so long as all the variables have their proper values for that point.

(Let the student determine how the conditions of this motion could be approximated to experimentally.)

83. Relative and Absolute Velocities.—Fig. 98. Let M be a material point having a uniform motion of velocity v_2 along a straight groove cut in the deck of a steamer, which itself has a uniform motion of translation, of velocity v_1, over the bed of a river. In one second M advances a distance v_2 along the groove, which simultaneously has moved a distance $v_1 = AB$ with the vessel. The absolute path of M during the second is evidently

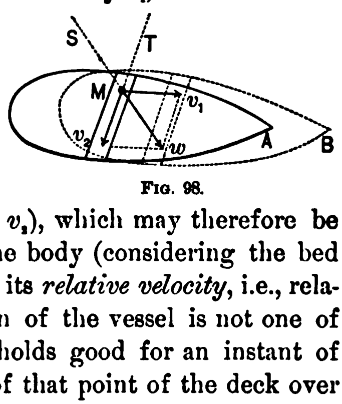


FIG. 98.

w (the diagonal formed on v_1 and v_2), which may therefore be called the *absolute velocity* of the body (considering the bed of the river as fixed); while v_2 is its *relative velocity*, i.e., relative to the vessel. If the motion of the vessel is not one of translation, the construction still holds good for an instant of time, but v_1 is then the velocity of that point of the deck over which M is passing at this instant, and v_2 is M's velocity relatively to that point alone.

Conversely, if M be moving over the deck with a given absolute velocity $= w$, v_1 being that of the vessel, the relative velocity v_2 may be found by resolving w into two components, one of which shall be v_1; the other will be v_2.

If w is the absolute velocity and direction of the *wind*, the vane on the mast-head will be parallel to MT, i.e., to v_2 the relative velocity; while if the vessel be rolling ... st-head therefore describing a sinuous path, the ... vane varies periodically.

Evidently the effect of the wind on the sails, if any, will depend on v_2 the relative, and not directly on w the absolute, velocity. Similarly, if w is the velocity of a jet of water, and v_1 that of a water-wheel channel, which the water is to enter without sudden deviation, or impact, the channel-partition should be made tangent to v_2 and not to w.

Again, the *aberration of light* of the stars depends on the same construction; v_1 is the absolute velocity of a locality of the earth's surface (being practically equal to that of the centre); w is the absolute direction and velocity of the light from a certain star. To see the star, a telescope must be directed along MT, i.e., parallel to v_2 the relative velocity; just as in the case of the moving vessel, the groove must have the direction MT, if the moving material point, having an absolute velocity w, is to pass down the groove without touching its sides. Since the velocity of light $= 192{,}000$ miles per second $= w$, and that of the earth in its orbit $= 19$ miles per second $= v_1$, the angle of aberration SMT, Fig. 98, will not exceed 20 seconds of arc; while it is zero when w and v_1 are parallel.

Returning to the wind and sail-boat, it will be seen from Fig. 98 that when $v_1 =$ or even $> w$, it is still possible for v_2 to be of such an amount and direction as to give, on a sail properly placed, a small wind-pressure, having a small fore-and aft component, which in the case of our ice-boat may exceed the small fore-and-aft resistance of such a craft, and thus v_1 will be still further increased; i.e., an ice-boat may sometimes travel faster than the wind which drives it. This has often been proved experimentally on the Hudson River.

CHAPTER IV.

MOMENT OF INERTIA.

[Note.—For the propriety of this term and its use in Mechanics, see § 114; for the present we are only concerned with its geometrical nature.]

85. Plane Figures.—Just as in dealing with the centre of gravity of a plane figure (§ 23), we had occasion to sum the series $\int z dF$, z being the distance of any element of area, dF, from an axis; so in subsequent chapters it will be necessary to know the value of the series $\int z^2 dF$ for plane figures of various shapes referred to various axes. This summation $\int z^2 dF$ of the products arising from multiplying each elementary area of the figure by the *square* of its distance from an axis is called the **moment of inertia** *of the plane figure with respect to the axis in question;* its symbol will be I. If the axis is perpendicular to the plane of the figure, it may be named the *polar mom. of inertia* (§ 94); if the axis lies in the plane, the *rectangular* mom. of inertia (§§ 90–93). Since the I of a plane figure evidently consists of *four dimensions of length*, it may always be resolved into two factors, thus $I = Fk^2$, in which F = total area of the figure, while $k = \sqrt{I \div F}$, is called the **radius of gyration**, because if all the elements of area were situated at the *same* radial distance, k, from the axis, the moment of inertia would still be the same, viz.,

$$I = \int k^2 dF = k^2 \int dF = Fk^2.$$

86. Rigid Bodies.—Similarly, in dealing with the rotary motion of a rigid body, we shall [illegible] sum of the series $\int \rho^2 dM$, meaning the summation of [illegible] arising from multiplying the mass dM of each [illegible] dV of a

rigid body by the square of its distance from a specified axis. This will be called the *moment of inertia of the body with respect to the particular axis mentioned* (often indicated by a subscript), and will be denoted by I. As before, it can often be conveniently written Mk^2, in which M is the whole mass, and k its "radius of gyration" for the axis used, k being $= \sqrt{I \div M}$. If the body is *homogeneous*, the heaviness, γ, of all its particles will be the same, and we may write

$$I = \int \rho^2 dM = (\gamma \div g)\int \rho^2 dV = (\gamma \div g)\, Vk^2.$$

87. If the body is a homogeneous plate of an *infinitely small thickness* $= \tau$, and of area $= F$, we have $I = (\gamma \div g)\int \rho^2 dV = (\gamma \div g)\tau \int \rho^2 dF$; i.e., $= (\gamma \div g) \times$ *thickness* $\times$ *mom. inertia of the plane figure.*

88. Two Parallel Axes. Reduction Formula.—Fig. 99. Let Z and Z' be two parallel axes. Then $I_z = \int \rho^2 dM$, and $I_{z'} = \int \rho'^2 dM$. But d being the distances between the axes, so that $a^2 + b^2 = d^2$, we have $\rho'^2 = (x-a)^2 + (y-b)^2 = (x^2+y^2) + d^2 - 2ax - 2by$, and $\therefore$

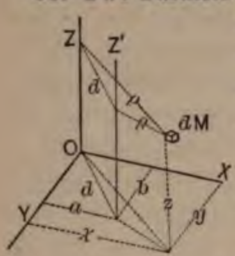


Fig. 99.

$$I_{z'} = \int \rho^2 dM + d^2 \int dM - 2a\int x dM - 2b\int y dM. \quad . \quad (1)$$

But $\int \rho^2 dM = I_z$, $\int dM = M$, and from the theory of the centre of gravity (see § 23, eq. (1), knowing that $dM = \gamma dV \div g$, and $\therefore$ that $[\int \gamma dV] \div g = M$) we have $\int x dM = M\bar{x}$ and $\int y dM = M\bar{y}$; hence (1) becomes

$$I_{z'} = I_z + M(d^2 - 2a\bar{x} - 2b\bar{y}), \quad . \quad . \quad . \quad . \quad (2)$$

in which a and b are the x and y of the axis Z'; $\bar{x}$ and $\bar{y}$ refer to the centre of gravity of the body. If Z is a gravity-axis (call it g), both $\bar{x}$ and $\bar{y} = 0$, and (2) becomes

$$I_{z'} = I_g + Md^2. \; . \; . \; . \quad \text{or} \quad k_{z'}^2 = k_g^2 + d^2. \quad . \quad . \quad . \quad (3)$$

It is therefore evident that the mom. of inertia about a gravity-axis is smaller than about any other *parallel* axis.

Eq. (3) includes the particular case of a *plane figure* ...

writing area instead of mass, i.e., when Z (now g) is a gravity-axis,

$$I_{z'} = I_g + Fd^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

89. Other Reduction Formulæ; for Plane Figures.—(The axes here mentioned lie in the plane of the figure.) For *two sets of rectangular axes*, having the *same* origin, the following holds good. Fig. 100. Since

$$I_X = \int y^2 dF, \quad \text{and} \quad I_Y = \int x^2 dF,$$

we have $\qquad I_X + I_Y = \int (x^2 + y^2) dF.$

Similarly, $\qquad I_U + I_V = \int (v^2 + u^2) dF.$

But since the x and y of any dF have the same hypothenuse as the u and v, we have $v^2 + u^2 = x^2 + y^2$; $\therefore\ I_X + I_Y = I_U + I_V$.

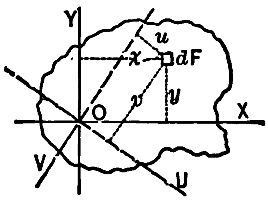


FIG. 100.

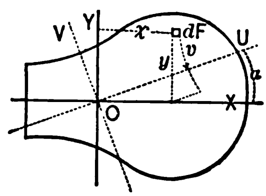


FIG. 100a.

Let X be an axis of symmetry; then, given I_X and I_Y (O is anywhere on X), required I_U, *U being an axis through O and making any angle α with X.*

$$I_U = \int v^2 dF = \int (y \cos \alpha - x \sin \alpha)^2 dF; \text{ i.e.,}$$
$$I_U = \cos^2 \alpha \int y^2 dF - 2 \sin \alpha \cos \alpha \int xy dF + \sin^2 \alpha \int x^2 dF.$$

But since the area is symmetrical about X, in summing up the products $xydF$, for every term $x(+y)dF$, there is also a term $x(-y)dF$ to cancel it; which gives $\int xydF = 0$. Hence

$$I_U = \cos^2 \alpha I_X + \sin^2 \alpha I_Y.$$

The student may easily prove that if two distances a and b be set off from O on X and Y respectively, made inversely proportional to $\sqrt{I_X}$ and $\sqrt{I_Y}$, and an ellipse described on a and b as semi-axes; then the moments of inertia of the figure about

any axes through O are inversely proportional to the squares of the corresponding semi-diameters of this ellipse; called therefore the *Ellipse of Inertia.* It follows therefore that the moments of inertia about *all gravity-axes* of a circle, or a regular polygon, are equal; since their ellipse of inertia must be a circle. Even if the plane figure is not symmetrical, an "ellipse of inertia" can be located at any point, and has the properties already mentioned; its axes are called the *principal axes* for that point.

90. The Rectangle.—*First, about its base.* Fig. 101. Since all points of a strip parallel to the base have the same co-ordinate, z, we may take the area of such a strip for $dF = bdz$;

FIG. 101.

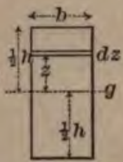


FIG. 102.

$$\therefore I_B = \int_0^h z^2 dF = b\int_0^h z^2 dz$$

$$= \tfrac{1}{3}b\Big[_0^h z^3 = \tfrac{1}{3}bh^3.$$

Secondly, about a *gravity-axis parallel to base.*

$$dF = bdz \therefore I_g = \int z^2 dF = b\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} z^2 dz = \tfrac{1}{12}bh^3.$$

Thirdly, about *any other axis in its plane.* Use the results already obtained in connection with the reduction-formulæ of §§ 88, 89.

90a. The Triangle.—*First*, about an axis through the vertex and parallel to the base; i.e., I_V in Fig. 103. Here the length of the strip is variable; call it y. From similar triangles

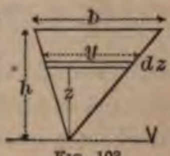


FIG. 103.

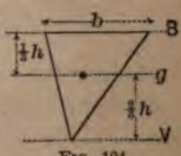


FIG. 104.

$$y = (b \div h)z;$$

$$\therefore I_V = \int z^2 dF = \int z^2 y dz = (b \div h)\int_0^h z^3 dx = \tfrac{1}{4}bh^3.$$

Secondly, about g, a gravity-axis parallel to the base. Fig. 104. From § 88, eq. (4), we have, since $F = \frac{1}{2}bh$ and

$$d = \tfrac{2}{3}h,\ I_g = I_V - Fd^2 = \tfrac{1}{4}bh^3 - \tfrac{1}{2}bh \cdot \tfrac{4}{9}h^2 = \tfrac{1}{36}bh^3.$$

Thirdly, Fig. 104, *about the base ;* $I_B = ?$ From § 88, eq. (4), $I_B = I_g + Fd^2$, with $d = \frac{1}{3}h$; hence

$$I_B = \tfrac{1}{36}bh^3 + \tfrac{1}{2}bh \,.\, \tfrac{1}{9}h^2 = \tfrac{1}{12}bh^3.$$

91. The Circle.—About any diameter, as g, Fig. 105. Polar co-ordinates, $I_g = \int z^2 dF$. Here we take $dF =$ area of an elementary rectangle $= \rho d\varphi . d\rho$, while $z = \rho \sin \varphi$.

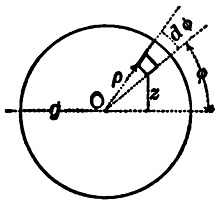


FIG. 105.

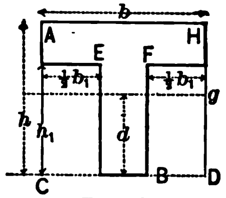


FIG. 106.

$$I_g = \int\int (\rho \sin \varphi)^2 \rho d\varphi d\rho = \int_0^{2\pi} \left[\sin^2 \varphi d\varphi \int_0^r \rho^3 d\rho \right]$$

$$= \frac{r^4}{4}\int_0^{2\pi} \sin^2 \varphi d\varphi = \frac{r^4}{4}\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2\varphi) d\varphi$$

$$= \frac{r^4}{4}\int_0^{2\pi} \left[\frac{1}{2} d\varphi - \frac{1}{4} \,.\, \cos 2\varphi d(2\varphi)\right]$$

$$= \frac{1}{4} r^4 \left[\left(\frac{1}{2}\varphi - \frac{1}{4} \sin 2\varphi\right) \right]_0^{2\pi}$$

$$= \frac{1}{4} r^4 \left[\left(\frac{2\pi}{2} - 0\right) - (0 - 0) \right]. \quad \text{i.e., } I_g = \frac{1}{4}\pi r^4.$$

92. Compound Plane Figures.—Since $I = \int z^2 dF$ is an infinite series, it may be considered as made up of separate groups or subordinate series, combined by algebraic addition, corresponding to the subdivision of the compound figure into component figures, each subordinate series being the moment of inertia of one of these component figures; but these separate moments *must all be referred to the same axis.* It is convenient to remember that the (*rectangular*) I of a plane figure remains unchanged if we conceive some or all of its elements shifted any distance parallel to the axis of reference. E.g., in Fig. 106, the sum of the I_B of the rectangle CE, and that of FD is $=$ to the I_B of the imaginary rectangle

formed by shifting one of them parallel to B, until it touches the other; i.e., I_B of $CE + I_B$ of $ED = \frac{1}{3}b_1h_1^3$ (§ 90). Hence the I_B of the $\top$ shape in Fig. 106 will be $= I_B$ of rectangle $AB - I_B$ of rect. $CE - I_B$ of rect. FD.

That is, I_B of $\top = \frac{1}{3}[bh^3 - b_1h_1^3]$. . . . (§ 90). . . (1)

About the gravity-axis, g, Fig. 106. To find the distance d from the base to the centre of gravity, we may make use of eq. (3) of § 23, writing areas instead of volumes, or, experimentally, having cut the given shape out of sheet-metal or card-board, we may balance it on a knife-edge. Supposing d to be known by some such method, we have, from eq. (4) of § 88, since the area $F = bh - b_1h_1$, $I_g = I_B - Fd^2$;

$$\text{i.e., } I_g = \tfrac{1}{3}[bh^3 - b_1h_1^3] - (bh - b_1h_1)d^2. \quad . \quad . \quad (2)$$

The double-$\top$ (*or* $\mathtt{I}$), and the *box* forms of Fig. 106*a*, *if symmetrical about the gravity-axis* g, have moments of inertia alike in form. Here the gravity-axis (parallel to base) of the compound figure is also a gravity axis (parallel to base) of each of the two component rectangles, of dimensions b and h, b_1 and h_1, respectively.

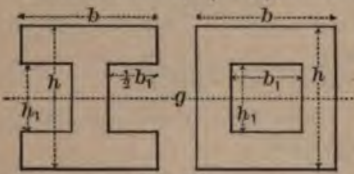


FIG. 106*a*.

Hence by algebraic addition we have (§ 90), for either compound figure,

$$I_g = \tfrac{1}{12}[bh^3 - b_1h_1^3]. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

(If there is no axis of symmetry parallel to the base we must proceed as in dealing with the $\top$-form.) Similarly for the ring,

FIG. 107.

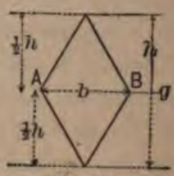


FIG. 108.

Fig. 107, or space between two *concentric* circumferences, we have, about any diameter or g (§ 91),

$$I_g = \tfrac{1}{4}\pi(r_1^4 - r_2^4). \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The rhombus about a gravity-axis, g, perpendicular to a diagonal, Fig. 108.—This axis divides the figure into two equal triangles, *symmetrically placed,* hence the I_g of the rhombus equals double the moment of inertia of one triangle about its base; hence (§ 90*a*)

$$I_g = 2 \cdot \tfrac{1}{12}b(\tfrac{1}{2}h)^3 = \tfrac{1}{48}bh^3. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

(The result is the same, if either vertex, or both, be shifted any distance parallel to AB.)

For practice, the student may derive results for the *trapezoid;* for the forms in Fig. 106, when the inner corners are rounded into equal quadrants of circles; for the double-T, when the lower flanges are shorter than the upper; for the regular polygons, etc.

93. If the plane figure be bounded, wholly or partially, by curves, it may be subdivided into an infinite number of strips, and the moments of inertia of these (referred to the desired axis) added by integration, *if the equations of the curves are known;* if not, Simpson's Rule, for a finite even number of strips, of equal width, may be employed for an approximate result. If these strips are parallel to the axis, the I of any one strip = its length × its width × square of distance from axis; while if perpendicular to, and *terminating in,* the axis, its $I = \frac{1}{3}$ its width × cube of its length (see § 90).

A graphic method of determining the moment of inertia of any irregular figure will be given in a subsequent chapter.

94. Polar Moment of Inertia of Plane Figures (§ 85).—Since the axis is now perpendicular to the plane of the figure, intersecting it in a point, O, the distances of the elements of area will all *radiate* from this point, and would better be denoted by ρ instead of z; hence, Fig. 109, $\int \rho^2 dF$ is the polar moment of inertia of any plane figure about a specified point O; this may be denoted by I_p. But $\rho^2 = x^2 + y^2$, for each dF; hence

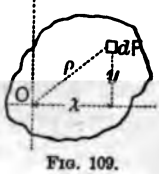


FIG. 109.

$$I_p = \int(x^2 + y^2)dF = \int x^2 dF + \int y^2 dF = I_Y + I_X.$$

i.e., *the polar moment of inertia about any given point in the plane equals the sum of the rectangular moments of inertia about any two axes of the plane figure, which intersect at right angles in the given point.* We have therefore for the *circle about its centre*

$$I_p = \tfrac{1}{4}\pi r^4 + \tfrac{1}{4}\pi r^4 = \tfrac{1}{2}\pi r^4;$$

For a *ring* of radii r_1 and r_2,

$$I_p = \tfrac{1}{2}\pi(r_1^4 - r_2^4);$$

For the *rectangle about its centre*,

$$I_p = \tfrac{1}{12}bh^3 + \tfrac{1}{12}hb^3 = \tfrac{1}{12}bh(b^2 + h^2);$$

For the *square*, this reduces to

$$I_p = \tfrac{1}{6}b^4.$$

(See §§ 90 and 91.)

95. Slender, Prismatic, Homogeneous Rod.—Returning to the moment of inertia of rigid bodies, or solids, we begin with that of a *material line*, as it might be called, about an axis through its extremity making some angle α with the rod. Let l = length of the rod, F its cross-section (very small, the result being strictly true only when $F = 0$). Subdivide the rod into an infinite number of small prisms, each having F as a base, and an altitude $= ds$. Let γ = the heaviness of the material; then the mass of an elementary prism, or $dM, = (\gamma \div g)Fds$, while its distance from the axis Z is $\rho = s \sin \alpha$. Hence the moment of inertia of the rod with respect to Z as an axis is

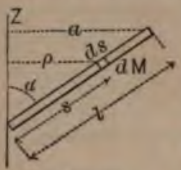


FIG. 110.

$$I_Z = \int \rho^2 dM = (\gamma \div g)F \sin^2 \alpha \int_0^l s^2 ds = \tfrac{1}{3}(\gamma \div g)Fl^3 \sin^2 \alpha.$$

But $\gamma Fl \div g$ = mass of rod and $l \sin \alpha = a$, the distance of the further extremity from the axis; hence $I_Z = \frac{1}{3}Ma^2$ and the *radius of gyration*, or k, is found by writing $\frac{1}{3}Ma^2 = Mk^2$; $\therefore k^2 = \frac{1}{3}a^2$, or $k = \sqrt{\frac{1}{3}}a$ (see § 86). If $\alpha = 90°$, $a = l$.

96. Thin Plates. Axis in the Plate.—Let the plates be homogeneous and of small constant thickness $= \tau$. If the surface of

the plate be $= F$, and its heaviness γ, then its mass $= \gamma F\tau \div g$. From § 87 we have for the plate, about any axis,

$I = (\gamma \div g)\tau \times$ *mom. of inertia of the plane figure formed by the shape of the plate.* . . . . (1)

Rectangular plate. Gravity-axis parallel to base.—Dimensions b and h. From eq. (1) and § 90 we have

$$I_g = (\gamma \div g)\tau \,.\, \tfrac{1}{12}bh^3 = (\gamma bh\tau \div g)\tfrac{1}{12}h^2 = \tfrac{1}{12}Mh^2; \ \therefore k^2 = \tfrac{1}{12}h^2.$$

Similarly, if *the base is the axis*, $I_B = \frac{1}{3}Mh^2$, $\therefore k^2 = \frac{1}{3}h^2$.

Triangular plate. Axis through vertex parallel to base.—From eq. (1) and § 90*a*, dimensions being b and h,

$$I_V = (\gamma \div g)\tau\tfrac{1}{4}bh^3 = (\gamma\tfrac{1}{2}bh\tau \div g)\tfrac{1}{2}h^2 = \tfrac{1}{2}Mh^2; \ \therefore k^2 = \tfrac{1}{2}h^2.$$

Circular plate, with any diameter as axis.—From eq. (1) and § 91 we have

$$I_g = (\gamma \div g)\tau\tfrac{1}{4}\pi r^4 = (\gamma\pi r^2\tau \div g)\tfrac{1}{4}r^2 = \tfrac{1}{4}Mr^2; \ \therefore k^2 = \tfrac{1}{4}r^2.$$

97. Plates or Right Prisms of any Thickness (or Altitude). Axis Perpendicular to Surface (or Base).—As before, the solid is homogeneous, i.e., of constant heaviness γ; let the altitude $= h$. Consider an elementary prism, Fig. 111, whose length is parallel to the axis of reference Z. Its altitude $= h =$ that of the whole solid; its base $= dF =$ an element of F the area of the base of solid; and each point of it has the same ρ. Hence we may take its mass, $= \gamma h dF \div g$, as the dM in summing the series

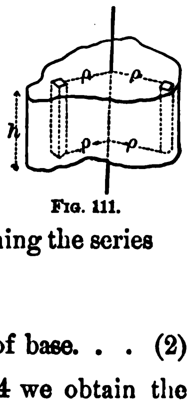


FIG. 111.

$$I_Z = \int \rho^2 dM;$$

$$\therefore I_Z = (\gamma h \div g)\int \rho^2 dF$$

$$= (\gamma h \div g) \times \text{polar mom. of inertia of base.} \quad . \ . \ (2)$$

By the use of eq. (2) and the results in § 94 we obtain the following:

Circular plate, or right circular cylinder, about the geometrical axis. $r =$ radius, $h =$ altitude.

$$I_g = (\gamma h \div g)\tfrac{1}{2}\pi r^4 = (\gamma h\pi r^2 \div g)\tfrac{1}{2}r^2 = \tfrac{1}{2}Mr^2; \ \therefore k^2 = \tfrac{1}{2}r^2.$$

Right parallelopiped or rectangular plate.—Fig. 112,

$$I_g = (\gamma h \div g)\tfrac{1}{12}bb_1(b_1^2 + b^2) = M\tfrac{1}{12}d^2; \ \therefore k^2 = \tfrac{1}{12}d^2.$$

For a *hollow cylinder*, about its geometric axis,

$$I_g = (\gamma \div g)\tfrac{1}{2}\pi(r_1^4 - r_2^4) = \tfrac{1}{2}M(r_2^2 + r_1^2);\ \therefore k^2 = \tfrac{1}{2}(r_2^2 + r_1^2).$$

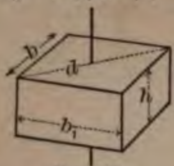


FIG. 112.

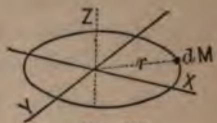


FIG. 113.

98. Circular Wire.—Fig. 113 (perspective). Let Z be a gravity-axis perpendicular to the plane of the wire; X and Y lie in this plane, intersecting at right angles in the centre O. The wire is homogeneous and of constant (small) cross-section. Since, referred to Z, each dM has the same $\rho = r$, we have $I_Z = \int r^2 dM = Mr^2$. Now I_X must equal I_Y, and (§ 94) their sum $= I_Z$,

$$\therefore I_X, \text{ or } I_Y, = \tfrac{1}{2}Mr^2, \qquad \text{and} \qquad k_X^2, \text{ or } k_Y^2 = \tfrac{1}{2}r^2.$$

99. Homogeneous Solid Cylinder, *about a diameter of its base.*—Fig. 114. $I_X = ?$ Divide the cylinder into an infinite number of laminæ, or thin plates, parallel to the base. Each is some distance z from X, of thickness dz, and of radius r (constant). In each draw a gravity-axis (of its own) parallel to X. We may now obtain the I_X of the whole cylinder by adding the I_X's of all the laminæ. The I_g of *any one lamina* (§ 96, circular plate) $=$ its mass $\times \frac{1}{4}r^2$; hence its I_X (eq. (3), § 88) $=$ its $I_g +$ (its mass) $\times z^2$. Hence for the whole cylinder

FIG. 114.

$$I_X = \int^h [(\gamma dz \pi r^2 \div g)(\tfrac{1}{4}r^2 + z^2)]$$

$$= (\pi r^2 \gamma \div g)\left[\tfrac{1}{4}r^2 \int_0^h dz + \int_0^h z^2 dz\right];$$

i.e., $I_X = (\pi r^2 h \gamma \div g)(\frac{1}{4}r^2 + \frac{1}{3}h^2) = M(\frac{1}{4}r^2 + \ldots$

100. Let the student prove (1) that if Fig. 114 ... **any right prism,** and k_F denote the radius of gyration ... one lamina, referred to its gravity-axis parallel to X, ... I_X of whole prism $= M(k_F^2 + \frac{1}{3}h^2)$; and (2) that the ...

of inertia of the cylinder about a gravity-axis parallel to the base is $= M(\frac{1}{4}r^2 + \frac{1}{12}h^2)$.

101. Homogeneous Right Cone.—Fig. 115. *First,* about an axis V, *through the vertex and parallel to* the base. As before, divide into laminæ parallel to the base. Each is a circular thin plate, but its radius, x, is not $= r$, but, from proportion, is $x = (r \div h)z$.

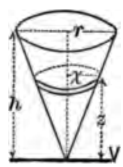


FIG. 115.

The I of any lamina referred to its own gravity-axis parallel to V is (§ 96) = (its mass) $\times \frac{1}{4}x^2$, and its I_V (eq. (3), § 88) is $\therefore$ = its mass $\times \frac{1}{4}x^2 +$ its mass $\times z^2$.

Hence for the whole cone,

$$I_V = \int_0^h (\pi x^2 dz\gamma \div g)[\tfrac{1}{4}x^2 + z^2]$$

$$= \frac{\gamma\pi r^2}{gh^2}\left[\frac{1}{4}\cdot\frac{r^2}{h^2} + 1\right]\int_0^h z^4 dz = M\frac{3}{20}[r^2 + 4h^2].$$

Secondly, about a gravity-axis parallel to the base.—From eq. (3), § 88, with $d = \frac{3}{4}h$ (see Prob. 7, § 26), and the result just obtained, we have $I = M\frac{3}{20}[r^2 + \frac{1}{4}h^2]$.

Thirdly, about its geometric axis, Z.—Fig. 116. Since the axis is perpendicular to each circular lamina through the centre, its I_Z (§ 97) is

$$= \text{its mass} \times \tfrac{1}{2}(\text{rad.})^2 = (\gamma\pi x^2 dz \div g)\tfrac{1}{2}x^2.$$

Now $x = (r \div h)z$, and hence for the whole cone

$$I_Z = \tfrac{1}{2}(\gamma\pi r^4 \div gh^4)\int_0^h z^4 dz = (\tfrac{1}{3}\pi r^2 h\gamma \div g)\tfrac{3}{10}r^2 = M\tfrac{3}{10}r^2.$$

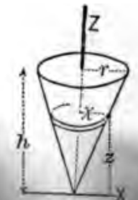


FIG. 116.

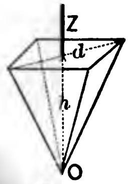


FIG. 117.

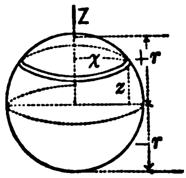


FIG. 118.

Homogen... Right **Pyramid of Rectangular Base.**—

...r. Proceeding as in the last para-

graph, we derive $I_Z = M\frac{1}{20}d^2$, in which d is the diagonal of the base.

103. Homogeneous Sphere.—About any diameter. Fig. 118. $I_Z = ?$ Divide into laminæ perpendicular to Z. By § 97, and noting that $x^2 = r^2 - z^2$, we have finally, for the whole sphere,

$$I_Z = (\gamma\pi \div 2g)\Big[_{-r}^{+r}(r^4z - \tfrac{2}{3}r^2z^3 + \tfrac{1}{5}z^5) = \tfrac{8}{15}\gamma\pi r^5 \div g$$
$$= (\tfrac{4}{3}\pi r^3\gamma \div g)\tfrac{2}{5}r^2 = M\tfrac{2}{5}r^2;\ \therefore k_z^2 = \tfrac{2}{5}r^2.$$

For a *segment*, of one or two bases, put proper limits for z in the foregoing, instead of $+r$ and $-r$.

104. Other Cases.—*Parabolic plate*, Fig. 119, homogeneous and of (any) constant thickness, about an axis through O, the middle of the chord, and perpendicular to the plate. This is

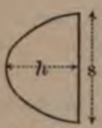


FIG. 119.

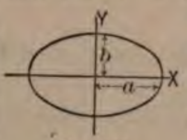


FIG. 120.

$$I = M\tfrac{1}{5}(\tfrac{1}{4}s^2 + \tfrac{8}{7}h^2).$$

The area of the segment is $= \frac{2}{3}hs$.

For an *elliptic plate*, Fig. 120, homogeneous and of any constant thickness, semi-axes a and b, we have about an axis through O, normal to surface $I_O = M\frac{1}{4}[a^2 + b^2]$; while for a very small constant thickness

$$I_X = M\tfrac{1}{4}b^2, \quad \text{and} \quad I_Y = M\tfrac{1}{4}a^2.$$

The area of the ellipse $= \pi ab$.

Considering Figs. 119 and 120 as *plane figures*, let the student determine their polar and rectangular moments of inertia about various axes.

(For still other cases, see p. 518 of Rankine's Applied Mechanics, and pp. 593 and 594 of Coxe's Weisbach.)

105. Numerical Substitution.—The *moments of inertia of plane figures* involve dimensions of length alone, and will be utilized in the problems involving flexure and torsion of beams, where the inch is the most convenient linear unit. E.g., the

polar moment of inertia of a circle of two inches radius about its centre is $\frac{1}{2}\pi r^4 = 25.13 +$ *biquadratic*, or *four-dimension*, inches, as it may be called. Since this quantity contains four dimensions of length, the use of the foot instead of the inch would diminish its numerical value in the ratio of the fourth power of twelve to unity.

The *moment of inertia of a rigid body, or solid,* however, $= Mk^2 = (G \div g)k^2$, in which G, the weight, is expressed in units of *force*, g involves both time and space (length), while k^2 involves length (two dimensions). Hence in any homogeneous formula in which the I of a solid occurs, we must be careful to employ units consistently; e.g., if in substituting $G \div g$ for M (as will always be done numerically) we put $g = 32.2$, we should use the *second* as unit of time, and the *foot* as linear unit.

106. Example.—Required the moment of inertia, about the axis of rotation, of a pulley consisting of a rim, four parallelopipedical arms, and a cylindrical hub which may be considered solid, being filled by a portion of the shaft. Fig. 121. Call the weight of the hub G, its radius r; similarly, for the rim, G_2, r_1 and r_2; the weight of one arm being $= G_1$. The total I will be the sum of the I's of the component parts, *referred to the same axis*, viz.: Those of the hub and rim will be $(G \div g)\frac{1}{2}r^2$ and $(G_2 \div g)\frac{1}{2}(r_1^2 + r_2^2)$, respectively (§ 97), while if the arms are *not very thick* compared with their length, we have for them (§§ 95 and 88)

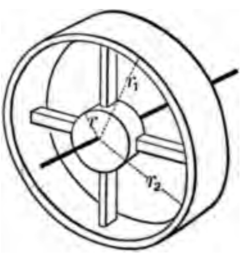


FIG. 121.

$$4(G \div g)\left[\tfrac{1}{3}(r_2 - r)^2 - \tfrac{1}{4}(r_2 - r)^2 + [r + \tfrac{1}{2}(r_2 - r)]^2\right],$$

as an approximation (obtained by reduction from the axis at the extremity of an arm to a parallel gravity-axis, then to the required [illegible] multiplying by four). In most fly-wheels, [illegible] heavy, besides being the farthest [illegible]n, that the moment of inertia [illegible]ical purposes neglected.

107. Ellipsoid of Inertia.—The moments of inertia about all axes passing through any given point of any rigid body whatever may be proved to be inversely proportional to the squares of the diameters which they intercept in an imaginary ellipsoid, whose centre is the given point, and whose position in the body depends on the distribution of its mass and the location of the given point. The three axes which contain the three principal diameters of the ellipsoid are called the *Principal Axes* of the body for the given point. This is called the **ellipsoid of inertia.** (Compare § 89.) Hence the moments of inertia of any homogeneous regular polyedron about all gravity-axes are equal, since then the ellipsoid becomes a sphere. It can also be proved that for any rigid body, if the co-ordinate axes X, Y, and Z, are taken coincident with the three principal axes at any point, we shall have

$$\int xydM = 0; \quad \int yzdM = 0; \quad \text{and} \quad \int zxdM = 0.$$

CHAPTER V.

DYNAMICS OF A RIGID BODY.

108. General Method.—Among the possible motions of a rigid body the most important for practical purposes (and fortunately the most simple to treat) are: a *motion of translation*, in which the particles move in parallel right lines with equal accelerations and velocities at any given instant; and *rotation about a fixed axis*, in which the particles describe circles in parallel planes with velocities and accelerations proportional (at any given instant) to their distances from the axis. Other motions will be mentioned later. To determine relations, or equations, between the elements of the motion, the mass and form of the body, and the forces acting (which do not necessarily form an unbalanced system), the most direct method to be employed is that of two *equivalent systems* of forces (§ 15), one consisting of the actual forces acting on the body, *considered free*, the other imaginary, consisting of the infinite number of forces which, applied to the separate material points composing the body, would account for their individual motions, as if they were an assemblage of particles without mutual actions or coherence. If the body were at rest, then considered *free*, and the forces referred to three co-ordinate axes, they would constitute a balanced system, for which the six summations ΣX, ΣY, ΣZ, $\Sigma(\text{mom.})_X$, $\Sigma(\text{mom.})_Y$, and $\Sigma(\text{mom.})_Z$, would each $= 0$; but in most cases of motion some or all of these sums are equal (at any given instant), not to zero, but to the corresponding summation of the imaginary equivalent system, i.e., to expressions involving the masses of the particles (or material points), their distribution in the body, and the

elements of the motion. That is, we obtain six equations by putting the ΣX of the actual system equal to the ΣX of the imaginary, and so on; for a definite instant of time (since some of the quantities may be variable).

109. Translation.—Fig. 122. At a given instant all the particles have the same velocity $= v$, in parallel right lines (parallel to the axis X, say), and the same acceleration p. Required the ΣX of the acting forces, shown at (I.). (II.) shows the imaginary equivalent system, consisting of a force = mass × acc. $= dMp$ applied parallel to X to each particle, since such a force would be necessary (from eq. (IV.) § 55) to account for the accelerated rectilinear motion of the particle, independently of the others. Putting $(\Sigma X)_I = (\Sigma X)_{II}$, we have

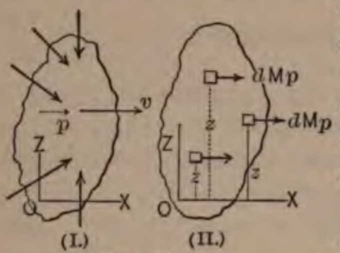


FIG. 122.

$$(\Sigma X)_I = \int p dM = p \int dM = Mp. \quad . \quad . \quad . \quad (\text{V.})$$

It is evident that the resultant of system (II.) must be parallel to X; hence that of (I.), which $= (\Sigma X)_I$ and may be denoted by R, must also be parallel to X; let a = perpendicular distance from R to the plane YX; a will be parallel to Z. Now put $[\Sigma(\text{mom.})_Y]_I = [\Sigma(\text{mom.}_Y)]_{II}$, ($Y$ is an axis perpendicular to paper through O) and we have $-Ra = -\int dMpz$ $= -p\int dMz = -pM\bar{z}$ (§ 88), i.e., $a = \bar{z}$. A similar result may be proved as regards $\bar{y}$. Hence, *if a rigid body has a motion of translation, the resultant force must act in a line through the centre of gravity* (here more properly called the centre of mass), *and parallel to the direction of motion.* Or, practically, in dealing with a rigid body having a motion of translation, we may consider it concentrated at its centre of mass. If the velocity of translation is uniform, $R = M \times 0$ $= 0$, i.e., the forces are balanced.

110. Rotation about a Fixed Axis.—First, as to the elements of space and time involved. Fig. 123. Let O be the axis of rotation (perpendicular to paper), OY a fixed line of reference, and OA a convenient line of the rotating body, passing through the axis and perpendicular to it, accompanying the body in its angular motion, which is the same as that of OA. Just as in linear motion we dealt with linear space (s), linear velocity (v), and linear acceleration (p), so here we distinguish at any instant;

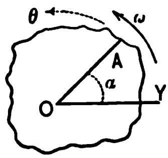


Fig. 123.

α, the *angular space* between OY and OA;

$\omega = \dfrac{d\alpha}{dt}$, the *angular velocity*, or rate at which α is changing;

and

$\theta = \dfrac{d\omega}{dt} = \dfrac{d^2\alpha}{dt^2}$, the *angular acceleration*, or rate at which ω is changing.

These are all reckoned in π-measure and may be $+$ or $-$, according to their direction against or with the hands of a watch.

(Let the student interpret the following cases: (1) at a certain instant ω is $+$, and θ $-$; (2) ω is $-$, and θ $+$; (3) α is $-$, ω and θ both $+$; (4) α $+$, ω and θ both $-$.) For rotary motion we have therefore, *in general*,

$$\omega = \frac{d\alpha}{dt}; \quad . \quad . \quad . \quad . \quad . \quad \text{(VI.)} \qquad \theta = \frac{d\omega}{dt} = \frac{d^2\alpha}{dt^2}; \quad . \quad . \quad \text{(VII.)}$$

and

$$\therefore \omega d\omega = \theta d\alpha; \quad . \quad . \quad . \quad . \quad . \quad \text{(VIII.)}$$

corresponding to eqs. (I.), (II.), and (III.) in § 50, for rectilinear motion.

Hence for *uniform rotary motion*, ω being constant and $\theta =$ [illegible] $= \omega t$, t being reckoned from the instant [illegible]

[illegible] *ry motion* θ is constant, and

if ω_0 denote the *initial* angular velocity (when α and $t = 0$), we may derive, precisely as in § 56,

$$\omega = \omega_0 + \theta t ; \quad . \quad . \quad (1) \qquad \alpha = \omega_0 t + \tfrac{1}{2}\theta t^2 ; \quad . \quad . \quad (2)$$

$$\alpha = \frac{\omega^2 - \omega_0^2}{2\theta} ; \quad . \quad . \quad (3) \quad \text{and} \quad \alpha = \tfrac{1}{2}(\omega_0 + \omega)t. \quad . \quad . \quad (4)$$

If in any problem in rotary motion θ, ω, and α have been determined for any instant, the corresponding *linear* values for any point of the body whose radial distance from the axis is ρ, will be $s = \alpha\rho$ (= distance described by the point measured along its circular path from its initial position), $v = \omega\rho$ = its velocity, and $p_t = \theta\rho$ its tangential acceleration, at the instant in question.

Examples.—(1) What value of ω, the angular velocity, is implied in the statement that a pulley is revolving at the rate of 100 revolutions per minute ?

100 revolutions per minute is at the rate of $2\pi \times 100$ $= 628.32$ (π-measure units) of angular space per minute $= 10.472$ per second ; $\therefore \omega = 628.32$ per minute or 10.472 per second.

(2) A grindstone whose initial speed of rotation is 90 revolutions per minute is brought to rest in 30 seconds, the angular retardation (or negative angular acceleration) being constant; required the angular acceleration, θ, and the angular space α described. Use the second as unit of time.

$\omega_0 = 2\pi\frac{90}{60} = 9.4248$ per second ; $\therefore$ from eq. (1)

$$\theta = \frac{\omega - \omega_0}{t} = -9.424 \div 30 = -0.3141 \ (\pi\text{-measure units})$$

per " square second." The angular space, from eq. (2) is

$$\alpha = \omega_0 t + \tfrac{1}{2}\theta t^2 = 30 \times 9.42 - \tfrac{1}{2}(0.314)900 = 141.3$$

(π-measure units), i.e., the stone has made 22.4 revolutions in coming to rest and a point 2 ft. from the axis has described a distance $s = \alpha\rho = 141.3 \times 2 = 282.6$ ft. in its circular path.

111. Rotation. Preliminary Problem. Axis Fixed.—For clearness in subsequent matter we now consider the following

simple case. Fig. 124 shows a rigid body, consisting of a drum, an axle, a projecting arm, all of which are *imponderable*, and a *single material point*, whose weight is G and mass M. An imponderable flexible cord, in which the tension is kept constant and $= P$, unwinds from the drum. The axle coincides with the vertical axis Z, while the cord

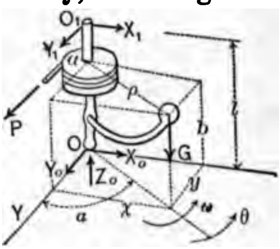


FIG. 124.

is always parallel to Y. Initially (i.e., when $t = 0$) M lies *at rest* in the plane ZY. Required its position at the end of any time t (i.e., at any instant) and also the reactions of the bearings at O and O_1, supposing no vertical pressure to exist at O_1, and that P and M are at the same level. No friction. At any instant the eight unknowns, α, ω, θ, X_0, Y_0, Z_0, X_1, and Y_1, may be found from the six equations formed by putting ΣX, etc., of the system of forces in Fig. 124, equal, respectively, to the ΣX, etc., of the imaginary equivalent system in Fig. 125, and two others to be mentioned subsequently. Since, at this instant, the velocity of M must be $v = \omega\rho$ and its tangential acceleration $p_t = \theta\rho$, its circular motion could be produced, considering it *free* (eq. (5), § 74), by a tangential force $T =$ mass $\times p_t = M\theta\rho$, and a normal centripetal force $N = Mv^2 \div \rho = M(\omega\rho)^2 \div \rho = \omega^2 M\rho$. Hence the system in Fig. 125 is equivalent

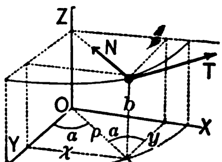


FIG. 125.

to that of Fig. 124, and from putting the $\Sigma\,(\text{mom.})_z$ of one = that of the other, we derive

$$Pa = T\rho;\ \text{i.e.,}\ Pa = \theta M\rho^2, \quad . \quad . \quad . \quad . \quad (1)$$

whence θ becomes known, and is evidently *constant*, since P, a, M, and ρ are such. ∴ the angular motion is *uniformly accelerated*, and from eqs. (1) and (2), § 110, ω and α become known;

i.e., $\omega = \theta t$, . . . (2) and $\alpha = \frac{1}{2}\theta t^2$. . . . . (3)

Putting (ΣZ of 124) = (ΣZ of 125 [illegible]

$$Z_0 - G = 0;\ \text{i.e.,}\ [illegible] \quad . \quad . \quad . \quad . \quad (4)$$

Proceeding similarly with the ΣX of each system,

$$X_0 + X_1 = T\cos\alpha - N\sin\alpha = \theta M\rho\cos\alpha - \omega^2 M\rho\sin\alpha, \quad (5)$$

and with the ΣY of each,

$$P + Y_0 + Y_1 = -T\sin\alpha - N\cos\alpha = -\theta M\rho\sin\alpha - \omega^2 M\rho\cos\alpha; \quad (6)$$

while with the Σ (mom.)$_X$ we have, conceiving all the forces in each system projected on the plane ZY (see § 38), and noting that $y = \rho\cos\alpha$ and $x = \rho\sin\alpha$,

$$+G\rho\cos\alpha + Y_1 l + Pb = -(\theta M\rho\sin\alpha)b - (\omega^2 M\rho\cos\alpha)b, \quad (7)$$

and with the Σ (mom.)$_Y$,

$$-G\rho\sin\alpha - X_1 l = -(\theta M\rho\cos\alpha)b + (\omega^2 M\rho\sin\alpha)b. \quad . \quad (8)$$

From (7) we may find Y_1; from (8), Y_1; then X_0 and Y_0 from (5) and (6). It will be noted that as the motion proceeds θ remains constant; ω increases with the time, α with the square of the time; Z_0 is constant, $= G$; while X_0, Y_0, X_1, and Z_1 have variable values dependent on $\rho\cos\alpha$ and $\rho\sin\alpha$, i.e., on the co-ordinates y and x of the moving material point.

112. Particular Supposition in the Preceding Problem with Numerical Substitution.—Suppose we have given (using the *foot-pound-second system* of units in which $g = 32.2$) $G = 64.4$ lbs., whence

$M = (G \div g) = 2$; $P = 4$ lbs., $l = 4$ ft., $b = 2$ ft., $a = 2$ ft.,

and $\rho = 4$ ft.; and that M is just passing through the plane ZX, i.e., that $\alpha = \frac{1}{2}\pi$. We obtain, first, the angular acceleration, eq. (1),

$$\theta = Pa \div M\rho^2 = 8 \div 32 = 0.25 = \tfrac{1}{4}.$$

From eqs. (2) and (3) we have *at the instant mentioned* (noting that when α was $= 0$, t was $= 0$)

$$\omega^2 = 2\alpha\theta = \tfrac{1}{4}\pi = 0.7854 +,$$

while (2) gives, for the time of describing the quadrant,

$$t = \omega \div \theta = 3.544 \ldots \text{ seconds.}$$

Since at this instant $\cos\alpha = 0$ and $\sin\alpha = 1$, we have, from (7),

$$-0 + Y_1 \times 4 + 4 \times 2 = -\tfrac{1}{4} \times 2 \times 4 \times 2; \therefore Y_1 = -3 \text{ lbs.}$$

The minus sign shows it should point in a direction contrary to that in which it is drawn in Fig. 124. Eq. (8) gives

$$-64.4\times 4 - X_1\times 4 = -0 + \tfrac{1}{4}\pi\times 2\times 4\times 4;\ \therefore X_1 = -70.683 \text{ lbs.}$$

And similarly, knowing Y_1 and X_1, we have from (5) and (6),

$$X_0 = +64.4 \text{ lbs.}, \quad \text{and} \quad Y_0 = -3.00 \text{ lbs.}$$

The resultant of X_1 and Y_1, also that of X_0, Y_0, and Z_0, can now be found by the parallelogram (and parallelopipedon) of forces, both in amount and position, noting carefully the directions of the components. These resultants are the actions of the supports upon the ends of the axle; *their equals and opposites* would be the actions or pressures of the axle against the supports, at the instant considered (when M is passing through the plane ZX; i.e., with $\alpha = \frac{1}{2}\pi$). (At the same instant, suppose *the string to break;* what would be the effect on the eight quantities mentioned?)

113. Centre of Percussion of a Rod suspended from one End.—Fig. 126. The rod is initially at rest (see (I.) in figure), is straight, homogeneous, and of constant (small) cross-section. Neglect its weight. A horizontal force or pressure, P, due to a blow (and varying in amount during the blow), now acts upon it from the left, perpendicularly to the axis, Z, of suspension. An accelerated

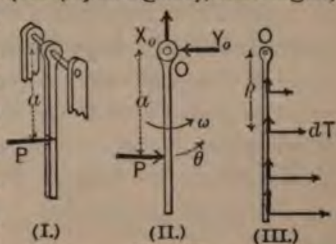


FIG. 126.

rotary motion begins about the fixed axis Z. (II.) shows the rod *free*, at a certain instant, with the reactions X_0 and Y_0 put in at O_0. (III.) shows an imaginary system which would produce the same effect at this instant, and consisting of a $dT = dM\theta\rho$, and a $dN = \omega^2 dM\rho$ applied to each dM, the rod being composed of an infinite number of dM's, each at some distance ρ from the axis. Considering that *the rotation has just begun*, ω, the angular velocity is as yet small, and will be neglected. Required Y, the horizontal reaction of the support at O in terms of P. By putting $\Sigma Y_{II} = \Sigma Y_{III}$, we have

$$P - \text{[illegible]} dM = \theta M\bar{\rho}.$$

$\therefore\ Y_0 = P - \theta M\bar{\rho}$; $\bar{\rho}$ is the distance of the centre of gravity from the axis (N.B. $\int \rho dM = M\bar{\rho}$ is only true when all the ρ's are in one plane or line as here). But the value of the angular acceleration θ at this instant depends on P and a, for $\Sigma\,(\text{mom.})_Z$ in (II.) $= \Sigma\,(\text{mom.})_Z$ in (III.), whence $Pa = \theta \int \rho^2 dM = \theta I_Z$, where I_Z is the *moment of inertia* of the rod about Z, and from § 95 $= \frac{1}{3}Ml^2$. Now $\bar{\rho} = \frac{1}{2}l$; hence, finally,

$$Y_0 = P\left[1 - \frac{3}{2} \cdot \frac{a}{l}\right].$$

If now Y_0 is to $= 0$, i.e., if there is to be *no shock between the rod and axis*, we need only apply P at a point whose distance $a = \frac{2}{3}l$ from the axis; for then $Y_0 = 0$. This point is called the **centre of percussion** for the given rod and axis. It and the point of suspension O are interchangeable (see § 118). (Lay a pencil on a table; tap it at a point distant one third of the length from one end; it will *begin to rotate* about a vertical axis through the farther end. Tap it at one end; it will begin to rotate about a vertical axis through the point first mentioned. Such an axis of rotation is called an *axis of instantaneous rotation*, and is different for each point of impact—just as the point of contact of a wheel and rail is the one point of the wheel which is momentarily at rest, and about which, therefore, all the others are turning *for the instant*. Tap the pencil at its centre of gravity, and a motion of translation begins; see § 109.)

114. Rotation. Axis Fixed. General Formulæ.—Consider-

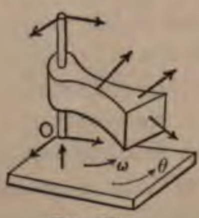


FIG. 127.

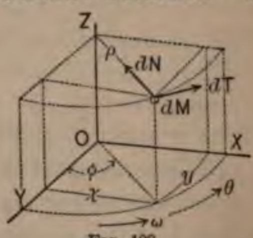


FIG. 128.

ing now a rigid body of any shape whatever, let Fig. 127 indicate the system of forces acting *at any given instant*, Z being

the fixed axis of rotation, ω and θ the angular velocity and angular acceleration, *at the given instant.* X and Y are two axes, at right angles to each other and to Z, fixed in space. At this instant each dM of the body has a definite x, y, and φ (see Fig. 128), which will change, and also a, ρ, and z, which will not change, as the motion progresses, and is pursuing a circular path with a velocity $= \omega\rho$ and a tangential acceleration $= \theta\rho$. Hence, if to each dM of the body (see Fig. 128) we imagine a tangential force $dT = dM\theta\rho$ and a normal force $= dM(\omega\rho)^2 \div \rho = \omega^2 dM\rho$ to be applied (eq. (5), § 74), and these alone, we have a system comprising an infinite number of forces, all parallel to XY, and equivalent to the actual system in Fig. 127. Let ΣX, etc., represent the sums (six) for Fig. 127, whatever they may be in any particular case, while for 128 we shall write the corresponding sums in detail. Noting that

$$\int dN \cos\varphi = \omega^2 \int dM\rho \cos\varphi = \omega^2 \int dMy = \omega^2 M\bar{y}, (\S\ 88);$$

that $\int dN \sin\varphi = \omega^2 \int dM\rho \sin\varphi = \omega^2 \int dMx = \omega^2 M\bar{x}$;

and similarly, that $\int dT \cos\varphi = \theta \int dM\rho \cos\varphi = \theta M\bar{y}$, and $\int dT \sin\varphi = \theta M\bar{x}$; while in the moment sums (the moment of $dT \cos\varphi$ about Y, for example, being $- dT \cos\varphi \,.\, z = - \theta dM\rho (\cos\varphi)z = -\theta dMyz$, the sum of the moms.$_Y$ of all the $(dT \cos\varphi)$'s $= - \theta \int dMyz$)

$$\int dT \cos\varphi z = \theta \int dMyz, \int dN \sin\varphi z = \omega^2 \int dMxz, \text{ etc.,}$$

we have, since the systems are equivalent,

$$\Sigma X = + \theta M\bar{y} - \omega^2 M\bar{x}; \quad . \quad . \quad . \quad . \quad (\text{IX.})$$

$$\Sigma Y = - \theta M\bar{x} - \omega^2 M\bar{y}; \quad . \quad . \quad . \quad . \quad (\text{X.})$$

$$\Sigma Z = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XI.})$$

$$\Sigma \text{ moms.}_X = - \theta \int dMxz - \omega^2 \int dMyz; \quad . \quad (\text{XII.})$$

$$\Sigma \text{ moms.}_Y = - \theta \int dMyz + \omega^2 \int dMxz; \quad . \quad (\text{XIII.})$$

$$\Sigma \text{ moms.}_Z = \theta \int dM\rho^2 = \theta I_Z. \quad . \quad . \quad . \quad (\text{XIV.})$$

These hold good for any instant. As the motion proceeds $\bar{x}$ and $\bar{y}$ change, as also the sums $\int dMxz$ and $\int dMyz$. [illegible] body, however, is homogeneous, and *symmetrical* [illegible] *plane* XY, $\int dMxz$ and $\int dMyz$ would always $=$ zero [illegible]

the z of any dM does not change, and for every term $dMy(+z)$, there would be a term $dMy(-z)$ to cancel it; similarly for $\int dMxz$. The eq. (XIV.), Σ (moms. about axis of rotat.) $= \int dT\rho = \theta \int dM\rho^2 =$ (*angular accel.*) $\times$ (*mom. of inertia of body about axis of rotat.*), shows how the sum $\int dM\rho^2$ arises in problems of this chapter. That a force $dT = dM\theta\rho$ should be necessary to account for the acceleration (tangential) $\theta\rho$ of the mass dM, is due to the so-called *inertia* of the mass (§ 54), and its moment $dT\rho$, or $\theta dM\rho^2$, might, with some reason, be called the *moment of inertia* of the dM, and $\int \theta dM\rho^2 = \theta \int dM\rho^2$ that of the whole body. But custom has restricted the name to the sum $\int dM\rho^2$, which, being without the θ, has no term to suggest the idea of inertia. For want of a better the name is still retained, however, and is generally denoted by I. (See §§ 86, etc.)

115. Example of the Preceding.—A homogeneous right parallelopiped is mounted on a vertical axle (no friction), as in figure. O is at its centre of gravity, *hence both* $\bar{x}$ *and* $\bar{y}$ *are zero.* Let its heaviness be γ, its dimensions h, b_1 and b (see § 97). XY is a plane of symmetry, hence both $\int dMxz$ and $\int dMyz$ are zero at all times (see above). The tension P in the (inextensible) cord is caused by the hanging weight P_1

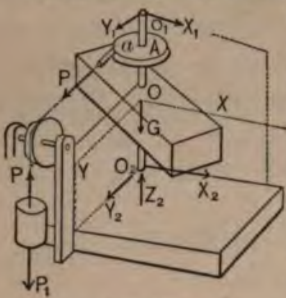


Fig. 120.

(but is not $= P_1$, unless the rotation is uniform). The figure shows both rigid bodies *free.* P_1 will have a motion of translation; the parallelopiped, one of rotation about a fixed axis. No masses are considered except $P_1 \div g$, and $bhb_1\gamma \div g$. The $I_Z = Mk_Z^2$ of the latter $=$ its mass $\times \frac{1}{12}(b_1^2 + b^2)$, § 97. At any instant, the cord being taut, if $p =$ linear acceleration of P_1, we have

$$p = \theta a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{eq. } (a)$$

From (XIV.), $\quad Pa = \theta I_Z; \;\; \therefore P = \theta I_Z \div a. \quad . \quad . \quad . \quad (1)$

For the free mass $P_1 \div g$ we have (§ 109) $P_1 - P =$ mass $\times$ acc.,

$$= (P_1 \div g)p = (P_1 \div g)\theta a\,; \; \therefore P = P_1(1 - \theta a \div g). \quad (2)$$

Equate these two values of P and solve for θ, whence

$$\theta = \frac{P_1 a}{Mk^2 + (P_1 \div g)a^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

All the terms here are *constant*, hence θ is constant; therefore the rotary motion is *uniformly accelerated*, as also the translation of P_1. The formulæ of § 56, and (1), (2), (3), and (5) of § 110, are applicable. The tension P is also constant; see eq. (1). As for the five unknown reactions (components) at O_1 and O_2, the bearings, we shall find that they too are constant; for

from (IX.) we have $\quad X_1 + X_2 = 0\,; \quad (4)$

from (X.) we have $\quad P + Y_1 + Y_2 = 0\,; \quad (5)$

from (XI.) we have $\quad Z_2 - G = 0\,; \quad (6)$

from (XII.) we have $\quad P\,.\,\overline{AO} + Y_1\,.\,\overline{O_1O} - Y_2\,.\,\overline{O_2O} = 0\,; \quad (7)$

from (XIII.) we have $\quad -X_1\,.\,\overline{O_1O} + X_2\,.\,\overline{O_2O} = 0. \quad (8)$

Numerical substitution in the above problem.—Let the parallelopiped be of wrought-iron; let $P_1 = 48$ lbs.; $a = 6$ in. $= \frac{1}{2}$ ft.; $b = 3$ in. $= \frac{1}{4}$ ft. (see Fig. 112); $b_1 = 2$ ft. 3 in. $= \frac{9}{4}$ ft.; and $h = 4$ in. $= \frac{1}{3}$ ft. Also let $\overline{O_1O} = \overline{O_2O} = 18$ in. $= \frac{3}{2}$ ft., and $\overline{AO} = 3$ in. $= \frac{1}{4}$ ft. Selecting the *foot-pound-second* system of units, in which $g = 32.2$, the linear dimensions must be used in feet, the heaviness, γ, of the iron must be used in *lbs. per cubic foot*, i.e., $\gamma = 480$ (see § 7), and all forces in lbs., times in seconds.

The weight of the iron will be $G = V\gamma = bb_1h\gamma = \frac{1}{4}\cdot\frac{9}{4}\cdot\frac{1}{3} \times 480 = 90$ lbs.; its mass $= 90 \div 32.2 = 2.79$; and its moment of inertia about $Z = I_Z = Mk_z^2 = M\frac{1}{12}(b_1^2 + b^2) = 2.79 \times 0.426 = 1.191$. (That is, the *radius of gyration*, k_z, $= \sqrt{0.426} = 0.653$ ft.; or the moment of inertia, or any result depending solely upon it, is just the same as if the mass were concentrated in a thin shell, or a line, or a point, at a distance of 0.653 feet from the axis.) We can now compute the angular acceleration, θ, from eq. (3);

$$\theta \quad \frac{48 \times \frac{1}{2}}{1.191 + (48 \div 32.2) \times \frac{1}{4}} = \frac{24}{1.191 + 0.372} = 15.36$$

π-measure units per "square second." The linear acceleration of P_1 is $p = \theta a = 7.68$ feet per square second for the uniformly accelerated translation.

Nothing has yet been said of the velocities and initial conditions of the motions; for what we have derived so far applies to any point of time. Suppose, then, that the angular velocity $\omega =$ zero when the time, $t = 0$; and correspondingly the velocity, $v = \omega a$, of translation of P_1, be also $= 0$ when $t = 0$. At the end of any time t, $\omega = \theta t$ (§§ 56 and 110) and $v = pt = \theta at$; also the angular space, $\alpha = \frac{1}{2}\theta t^2$, described by the parallelopiped during the time t, and the linear space $s = \frac{1}{2}pt^2 = \frac{1}{2}\theta at^2$, through which the weight P_1 has sunk vertically. For example, during the first second the parallelopiped has rotated through an angle $\alpha = \frac{1}{2}\theta t^2 = \frac{1}{2} \times 15.36 \times 1 = 7.68$ units, π-measure, i.e., $(7.68 \div 2\pi) = 1.22$ revolutions, while P_1 has sunk through $s = \frac{1}{2}\theta at^2 = 3.84$ ft., vertically.

The tension in the cord, from (2), is

$$P = 48(1 - 15.36 \times \tfrac{1}{2} \div g) = 48(1 - 0.24) = 36.48 \text{ lbs.}$$

The pressures at the bearings will be as follows, *at any instant:* from (4) and (8), X_1 and X_2 must individually be zero; from (6) $Z_2 = G = V\gamma = 90$ lbs.; while from (5) and (7), $Y_1 = Y_2 = -\frac{4}{5}P = -29.18$ lbs., and should point in a direction opposite to that in which they were assumed in Fig. 129 (see last lines of § 39).

116. Torsion Balance. A Variably Accel. Rotary Motion. Axis Fixed.—A homogeneous solid having an axis of symmetry is suspended by an elastic prism, or filament (whose mass may be neglected), so that the latter is vertical and coincident with the axis of symmetry, and is not only supported, but prevented from turning at its upper extremity. If the solid is turned about its axis away from its position of rest and set free, the torsional

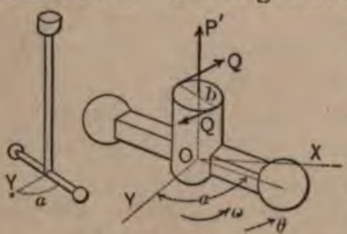


FIG. 130.

elasticity of the rod or filament, which is fixed in the solid, causes an oscillatory rotary motion. Required the duration of an oscillation. Fig. 130.

Take the axis Y at the middle of the oscillation (the original position of rest). Reckon the time from the instant of passing this position. Let the initial angular velocity $= \omega_0$. As the motion progresses ω diminishes, i.e., θ is negative.

To consider the body *free*, conceive the rod cut close to the body (in which it is *firmly inserted*), and in the section thus exposed put in the vertical tension P', and also the horizontal forces forming a couple to which at any instant the twisting action (of the portion of rod removed upon the part left in the free body) is known to be due. Call the moment of this couple Qb (known as the *moment of torsion*); it is variable, being directly proportional to the angle α; hence, if by experiment it is found to be $= Q_1b_1$ when α is $= \alpha_1$, for any value of α it will be $Qb = (Q_1b_1 \div \alpha_1)\alpha = C\alpha$, in which C is the constant factor.

At any instant, therefore, the forces acting are G, P', and those equivalent to the couple whose moment $= Qb = C\alpha$. (No lateral support is required; the student would find the X_1, Y_1, X_2, and Y_2 of Fig. 129 to be individually zero, if put in; remembering that here, $\bar{x}$ and $\bar{y}$ both $= 0$, as also $\int dMxz$ and $\int dMyz$; and that the forces of the couple will not be represented in any of the six summations of § 114, except in Σ moms.$_Z$)

From eq. (XIV.), § 114, we have $- Qb$, i.e., $- C\alpha, = \theta I_Z$, from which

$$\theta = -(C \div I_Z)\alpha, \text{ or, for short, } \theta = -B\alpha. \quad . \quad . \quad (1)$$

Since B is constant, and there is an initial (angular) velocity $= \omega_0$, and since the variables θ, ω, and α, in angular motion correspond precisely to those (p, v, and s) of rectilinear motion, it is evident that the p[illegible]nt is a case of *harmonic motion*, already discussed in Pro[illegible] of § 59. Applying the results there obtained, since B [illegible] corresponds to the a of that [illegible] are *isochronal*, i.e., their

durations are the same whatever the amplitude (provided the elasticity of the rod is not impaired), and that the duration of one oscillation (from one extreme position to the other) is $t' = \pi \div \sqrt{B}$, or finally,

$$t' = \pi \sqrt{\alpha_1 I_Z \div Q_1 b_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

117. The Compound Pendulum is any rigid body allowed to oscillate without friction under the action of gravity when mounted on a horizontal axis. Fig. 131 shows the body *free*, in any position during the progress of the oscillation. C is the centre of gravity; let $\overline{OC} = s$. From (XIV.), § 114, we have Σ (mom. about fixed axis)

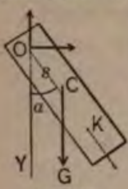


FIG. 131.

= angul. acc. × mom. of inertia.

$$\therefore - Gs \sin \alpha = \theta I_0,$$
$$\text{and } \theta = - Gs \sin \alpha \div I_0 = - Mgs \sin \alpha \div Mk^2,$$
$$\text{i.e., } \theta = - gs \sin \alpha \div k_0^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Hence θ is variable, proportional to sin α. Let us see what the length $l = \overline{OK}$, of a simple circular pendulum, must be, to have at this instant (i.e., for this value of α) the same angular acceleration as the rigid body. The linear (tangential) accelerations of K, the extremity of the required simple pendulum would be (§ 77) $p_t = g \sin \alpha$, and hence its angular acceleration would $= g \sin \alpha \div l$. Writing this equal to θ in eq. (1), we obtain

$$l = k_0^2 \div s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But this is *independent of* α; therefore the length of the simple pendulum having an angular acceleration equal to that of the oscillating body is *the same in all positions of the latter*, and if the two begin to oscillate simultaneously from a position of rest at any given angle α_1 with the vertical, they will keep abreast of each other during the whole motion, and hence have

the same duration of oscillation; which is $\therefore$, *for small amplitudes* (§ 78),

$$t' = \pi \sqrt{l \div g} = \pi \sqrt{k_0^2 \div gs}, \quad . \quad . \quad . \quad . \quad (3)$$

K is called the *centre of oscillation* corresponding to the given *centre of suspension* O, and is identical with the *centre of percussion* (§ 113).

Example.—Required the time of oscillation of a cast-iron cylinder, whose diameter is 2 in. and length 10 in., if the axis of suspension is taken 4 in. above its centre. If we use 32.2 for g, all linear dimensions should be in feet and times in seconds. From § 100, we have

$$I_C = M(\tfrac{1}{4}r^2 + \tfrac{1}{12}h^2) = M(\tfrac{1}{4} \cdot \tfrac{1}{144} + \tfrac{1}{12} \cdot \tfrac{100}{144}) = M\tfrac{1}{144} \cdot \tfrac{103}{12}.$$

From eq. (3), § 88,

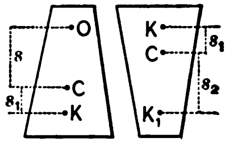
$$I_0 = I_C + Ms^2 = M[\tfrac{1}{144} \cdot \tfrac{103}{12} + \tfrac{1}{9}] = M \times 0.170;$$

$$\therefore k_0^2 = 0.170 \text{ sq. ft.}; \ \therefore t' = \pi \sqrt{0.170 \div 32.2 \times \tfrac{1}{3}} = 0.395 \text{ sec.}$$

118. The Centres of Oscillation and Suspension are Interchangeable.—(Strictly speaking, these centres are points in the line through the centre of gravity perpendicular to the axis of suspension.) Refer the centre of oscillation K to the centre of gravity, thus (Fig. 132, at (I.)):

$$s_1 = l - s = \frac{Mk_0^2}{Ms} - s = \frac{Mk_C^2 + Ms^2}{Ms} - s = \frac{k_C^2}{s}. \quad (1)$$

Now invert the body and suspend it at K; required CK_1, or s_2, to find the centre of oscillation corresponding to K as centre of suspension. By analogy from (1) we have $s_2 = k_C^2 \div s_1$; but from (1), $k_C^2 \div s_1 = s \therefore$ $s_2 = s$; in other words, K_1 is identical with O. Hence the proposition is proved.



(I.) (II.)

FIG. 132.

Advantage may be taken of this to determine the length L of the theoretical simple pendulum vibrating seconds, and thus finally the acceleration of gravity from formula (3), § 117, viz.,

when $t' = 1.0$ and l (now $= L$) has been determined experimentally, we have

$$g \text{ (in ft. per sq. second)} = L \text{ (in ft.)} \div \pi^2. \quad . \quad . \quad (2)$$

This most accurate method of determining g at any locality requires the use of a bar of metal, furnished with a sliding weight for shifting the centre of gravity, and with two projecting blocks provided with knife-edges. These blocks can also be shifted and clamped. By suspending the bar by one knife-edge on a proper support, the duration of an oscillation is computed by counting the total number in as long a period of time as possible; it is then reversed and suspended on the other with like observations. By shifting the blocks between successive experiments, the duration of the oscillation in one position is made the same as in the other, i.e., the distance between the knife-edges is the length, l, of the simple pendulum vibrating in the computed time (if the knife-edges are not equidistant from the centre of gravity), and is carefully measured. The l and t' of eq. (3), § 117, being thus known, g may be computed. Professor Bartlett gives as the length of the simple pendulum vibrating seconds at any latitude β

$$L \text{ (in feet)} = 3.26058 - 0.008318 \cos 2\beta.$$

119. Isochronal Axes of Suspension.—*In any compound pendulum, for any axis of suspension, there are always three others, parallel to it in the same gravity-plane, for which the oscillations are made in the same time as for the first.* For any assigned time of oscillation t', eq. (3), § 117, compute the corresponding distance $\overline{CO} = s$ of O from C;

i.e., from $$t'^2 = \pi^2 \frac{Mk_o^2}{Mgs} = \frac{\pi^2(Mk_C^2 + Ms^2)}{Mgs},$$

we have $$s = (gt'^2 \div 2\pi^2) \pm \sqrt{(g^2t'^4 \div 4\pi^4) - k_C^2}. \quad . \quad . \quad (1)$$

Hence for a given t', there are two positions for the axis O parallel to any axis through C, in any gravity-plane, on both sides; i.e., *four parallel axes of suspension*, in any gravity-plane, giving equal times of vibration; for two of these axes

we must reverse the body. E.g., if a slender, homogeneous, prismatic rod be marked off into thirds, the (small) vibrations will be of the same duration, if the centre of suspension is taken at either extremity, or at either point of division.

Example.—Required the positions of the axes of suspension, parallel to the base, of a right cone of brass, whose altitude is six inches, radius of base, 1.20 inches, and weight per cubic inch is 0.304 lbs., so that the time of oscillation may be a half-second. (N.B. For variety, use the inch-pound-second system of units, first consulting § 51.)

120. The Fly-Wheel in Fig. 133 at any instant experiences a pressure P' against its crank-pin from the connecting-rod and a resisting pressure P'' from the teeth of a spur-wheel with

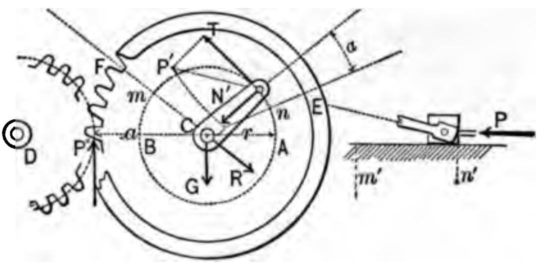


FIG. 133.

which it gears. Its weight G acts through C (nearly), and there are pressures at the bearings, but these latter and G have no moments about the axis C (perpendicular to paper). The figure shows it *free*, P'' being assumed constant (in practice this depends on the resistances met by the machines which D drives, and the fluctuation of velocity of their moving parts). P', and therefore T its tangential component, are variable, depending on the effective steam-pressure on the piston at any instant, on the obliquity of the connecting-rod, and in high-speed engines on the masses and motions of the piston and connecting-rod. Let $r =$ radius of crank-pin circle, and a the perpendicular from C on P''. From eq. (XIV.), § 114, we have

$$Tr - P''a = \theta I_C, \therefore \theta = (Tr - P''a) \div I_C \quad . \quad (1$$

as the angular acceleration at any instant; substituting which in the general equation (VIII.), § 110, we obtain

$$I_C \omega d\omega = Trd\alpha - P''ad\alpha. \quad . \quad . \quad . \quad . \quad (2)$$

From (1) it is evident that if at any position of the crank-pin the variable Tr is equal to the constant $P''a$, θ is zero, and consequently the angular velocity ω is either a maximum or a minimum. Suppose this is known to be the case both at m and n; i.e., suppose T, which was zero at the dead-point A, has been gradually increasing, till at n, $Tr = P''a$; and thereafter increases still further, then begins to diminish, until at m Tr again $= P''a$, and continues to diminish toward the dead-point B. The angular velocity ω, whatever it may have been on passing the dead-point A, diminishes, since θ is negative, from A to n, where it is ω_n, a minimum; increases from n to m, where it reaches a maximum value, ω_m. n and m being known points, and supposing ω_n known, let us inquire what ω_m will be. From eq. (2) we have

$$I_C \int_{\omega_n}^{\omega_m} \omega d\omega = \int_n^m Trd\alpha - P'' \int_n^m ad\alpha. \quad . \quad . \quad (3)$$

But $rd\alpha = ds' =$ an element of the path of the crank-pin, and also the "virtual velocity" of the force T, and $ad\alpha = ds''$, an element of the path of a point in the pitch-circle of the fly-wheel, the small space through which P'' is overcome in dt. Hence (3) becomes

$$I_C \tfrac{1}{2}(\omega_m^2 - \omega_n^2) = \int_n^m Tds - P'' \times \text{linear arc } \overline{EF}. \quad (4)$$

To determine $\int_n^m Tds$ we might, by a knowledge of the varying steam-pressure, the varying obliquity of the connecting-rod, etc., determine T for a number of points equally spaced along the curve nm, and obtain an approximate value of this sum by Simpson's Rule; but a simpler method is possible by noting (see eq. (1), § 65) that each term Tds of this sum = the corresponding term Pdx in the series $\int_{n'}^{m'} Pdx$, in which $P =$ the

effective steam-pressure on the piston in the cylinder at any instant, dx the small distance described by the piston while the crank-pin describes any ds, and n' and m' the positions of the piston (or of cross-head, as in Fig. 133) when the crank-pin is at n and m respectively. (4) may now be written

$$I_C\tfrac{1}{2}(\omega_m^2 - \omega_n^2) = \int_{n'}^{m'} Pdx - P'' \times \text{linear arc } \overline{EF}, \quad (5)$$

from which ω_m may be found as proposed. More generally, it is available, alone (or with other equations), to determine any one (or more, according to the number of equations) unknown quantity. This problem, in rotary motion, is analogous to that in § 59 (Prob. 4) for rectilinear motion. Friction and the inertia of piston and connecting-rod have been neglected. As to the time of describing the arc nm, from equations similar to (5), we may determine values of ω for points along nm, dividing it into an even number of equal parts, calling them ω_1, ω_2, etc., and then employ Simpson's Rule for an approximate value of the sum $\Big[_n^m t = \int_n^m \frac{d\alpha}{\omega}$ (from eq. (VI.), § 110); e.g., with four parts, we would have

$$\Big[_n^m t = \frac{1}{12}(\text{angle } nCm, \ \pi\text{-meas.})\left[\frac{1}{\omega_n} + \frac{4}{\omega_1} + \frac{2}{\omega_2} + \frac{4}{\omega_3} + \frac{1}{\omega_m}\right]. (6)$$

121. Numerical Example. Fly-Wheel.—(See Fig. 133 and the equations of § 120.) Suppose the engine is non-condensing and non-expansive (i.e., that P is constant), and that

$$P = 5500 \text{ lbs.}, \quad r = 6 \text{ in.} = \tfrac{1}{2} \text{ ft.}, \quad a = 2 \text{ ft.},$$

and also that the wheel is to make 120 revolutions per minute, i.e., that its *mean angular velocity* is to be

$$\omega' = \tfrac{120}{60} \times 2\pi, \text{ i.e., } \omega' = 4\pi.$$

First, required the amount of the resistance P'' (constant) that there shall be no permanent change of speed, i.e., that the angular velocity shall have the same value at the [illegible] complete revolution as at the beginning. Since a[illegible] form of eq. (5) holds good for any r[illegible]

that range be a complete revolution, and we shall have zero as the left-hand member; $\int P dx = P \times 2$ ft. $= 5500$ lbs. $\times 2$ ft., or 11,000 foot-pounds (as it may be called); while P'' is unknown, and instead of lin. arc $\overline{EF}$ we have a whole circumference of 2 ft. radius, i.e., 4π ft.;

$\therefore 0 = 11{,}000 - P'' \times 4 \times 3.1416$; whence $P'' = 875$ lbs.

Secondly, required the proper mass to be given to the fly-wheel of 2 ft. radius that in the forward stroke (i.e., while the crank-pin is describing its *upper* semicircle) the max. angular velocity ω_m shall exceed the minimum ω_n by only $\frac{1}{10}\omega'$, assuming (which is nearly true) that $\frac{1}{2}(\omega_m + \omega_n) = \omega'$. There being now three unknowns, we require three equations, which are, including eq. (5) of § 120, viz.:

$$Mk_C^2 \tfrac{1}{2}(\omega_m + \omega_n)(\omega_m - \omega_n) = \int_{n'}^{m'} P dx - P'' \times \text{linear arc } \overline{EF}; \quad (5)$$

$$\tfrac{1}{2}(\omega_m + \omega_n) = \omega' = 4\pi; \quad (7) \quad \text{and} \quad \omega_m - \omega_n = \tfrac{1}{10}\omega' = \tfrac{2}{5}\pi. \quad (8)$$

The points n and m are found most easily and with sufficient accuracy by a graphic process. Laying off the dimensions to scale, by trial such positions of the crank-pin are found that T, the tangential component of the thrust P' produced in the connecting-rod by the steam-pressure P (which may be resolved into two components, along the connecting-rod and a normal to itself) is $=(a \div r)P''$, i.e., is $= 3500$ lbs. These points will be n and m (and two others on the lower semicircle). The positions of the piston n' and m', corresponding to n and m of the crank-pin, are also found graphically in an obvious manner. We thus determine the angle nCm to be 100°, so that linear arc $EF = \frac{100}{180}\pi \times 2$ ft. $= \frac{10}{9}\pi$, while

$$\int_{n'}^{m'} P dx = 5500 \text{ lbs.} \times \int_{n'}^{m'} dx = 5500 \times \overline{n'm'} = 5500 \times .077 \text{ ft.,}$$

$n'm'$ being scaled from the draft.

Now substitute from (7) and (8) in (5), and we have, with $k_C = 2$ ft. (which assumes that the mass of the fly-wheel is concentrated in the rim),

$$(G \div g) \times 4 \times 4\pi \times \tfrac{2}{3}\pi = 5500 \times .077 - 875 \times \tfrac{10}{9}\pi,$$

which being solved for G (with $g = 32.2$; since we have used the foot and second), gives $G = 600.7$ lbs.

The points of max. and min. angular velocity on the back-stroke may be found similarly, and their values for the fly-wheel as now determined; they will differ but slightly from the ω_m and ω_n of the forward stroke. Professor Cotterill says that the rim of a fly-wheel should never have a max. velocity $>$ 80 ft. per sec.; and that if made in segments, not more than 40 to 50 feet per second. In the present example we have for the forward stroke, from eqs. (7) and (8), $\omega_m = 13.2$ (π-measure units) per second; i.e., the corresponding velocity of the wheel-rim is $v_m = \omega_m a = 26.4$ feet per second.

122. Angular Velocity Constant. Fixed Axis.—If ω is constant, the angular acceleration, θ, must be $=$ zero at all times, which requires Σ(mom.) about the axis of rotation to be $= 0$ (eq. (XIV.), § 114). An instance of this occurs when the only forces acting are the reactions at the bearings on the axis, and the body's weight, parallel to or intersecting the axis; the values of these reactions are now to be determined for different forms of bodies, in various positions relatively to the axis. (The opposites and equals of these reactions, i.e., the forces with which the axis acts upon the bearings, are sometimes stated to be due to the "*centrifugal forces*," or "centrifugal action," of the revolving body.)

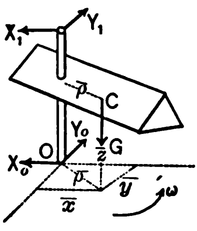


FIG. 134.

Take the axis of rotation for Z, then, with $\theta = 0$, the equations of § 114 reduce to

$$\Sigma X = -\omega^2 M\bar{x}; \quad . \quad . \quad . \quad . \quad \text{(IX}a.\text{)}$$
$$\Sigma Y = -\omega^2 M\bar{y}; \quad . \quad . \quad . \quad . \quad \text{(X}a.\text{)}$$
$$\Sigma Z = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XI}a.\text{)}$$
$$\Sigma \text{ moms.}_X = -\omega^2 \int dMyz; \quad . \quad . \quad . \quad \text{(XII}a.\text{)}$$
$$\Sigma \text{ moms.}_Y = +\omega^2 \int dMxz; \quad . \quad . \quad . \quad \text{(XIII}a.\text{)}$$
$$\Sigma \text{ moms.}_Z = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIV}$$

For greater convenience, let us suppose the axes X and Y (since their position is arbitrary so long as they are perpendicular to each other and to Z) *to revolve with the body* in its uniform rotation.

122a. *If a homogeneous body have a plane of symmetry and rotate uniformly about any axis Z perpendicular to that plane (intersecting it at O), then the acting forces are equivalent to a single force, $= \omega^2 M\bar{\rho}$, applied at O and acting in a gravity-line, but directed away from the centre of gravity.* It is evident that such a force $P = \omega^2 M\bar{\rho}$, applied as stated (see Fig. 135), will satisfy all six conditions expressed in the foregoing equations, taking X through the centre of gravity, so that $\bar{x} = \bar{\rho}$. For, from (IX*a*.), P must $= \omega^2 M\bar{\rho}$, while in each of the other summations the left-hand member will be zero, since P lies in the axis of X; and as their right-hand members will also be zero for the present body ($\bar{y} = 0$; and each of the sums $\int dMyz$ and $\int dMxz$ is zero, since for each term $dMy(+z)$ there is another $dMy(-z)$ to cancel it; and similarly, for $\int dMxz$), they also are satisfied; Q.E.D. Hence a single point of support at O will suffice to maintain the uniform motion of the body, and the pressure against it will be equal and opposite to P.

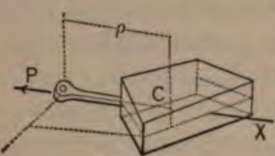


FIG. 135.

First Example.—Fig. 136. Supposing (for greater safety) that the uniform rotation of 210 revolutions per minute of each segment of a fly-wheel is maintained solely by the tension in the corresponding arm, P; required the value of P if the segment and arm together weigh $\frac{1}{30}$ of a ton, and the distance of their centre of gravity from the axis is $\bar{\rho} = 20$ in., i.e., $= \frac{5}{3}$ ft. With the *foot-ton-second* system of units, with $g = 32.2$, we have

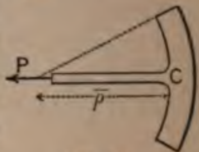


FIG. 136.

$$P = \omega^2 M\bar{\rho} = [\tfrac{210}{60} \times 2\pi]^2 \times [\tfrac{1}{30} \div 32.2] \times \tfrac{5}{3} = 0.83 \text{ tons},$$

or 1660 lbs.

Second Example.—Fig. 137. Suppose the uniform rotation of the same fly-wheel depends solely on the tension in the rim, required its amount. The figure shows the half-rim free, with the two equal tensions, P', put in at the surfaces exposed. Here it is assumed that the arms exert no tension on the rim. From § 122*a* we have $2P' = \omega^2 M\bar{\rho}$, where M is the mass of the half-rim, and $\bar{\rho}$ its gravity co-ordinate, which may be obtained approximately by § 26, Problem 1, considering the rim as a circular wire, viz., $\bar{\rho} = 2r \div \pi$.

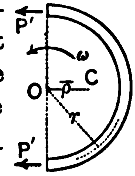


FIG. 137.

Let $M = (180 \text{ lbs.}) \div g$, with $r = 2$ ft. We have then

$$P' = \tfrac{1}{2}(22)^2(180 \div 32.2)(4 \div \pi) = 2033 \text{ lbs.}$$

(In reality neither the arms nor the rim sustain the tensions just computed; in treating the arms we have supposed no duty done by the rim, and *vice versâ*. The actual stresses are less, and depend on the yielding of the parts. Then, too, we have supposed the wheel to take no part in the transmission of motion by belting or gearing, which would cause a bending of the arms, and have neglected its weight.)

122b. *If a homogeneous body have a line of symmetry and rotate uniformly about an axis parallel to it (O being the foot of the perpendicular from the centre of gravity on the axis), then the acting forces are equivalent to a single force* $P = \omega^2 M\bar{\rho}$, *applied at O and acting in a gravity-line away from the centre of gravity.*

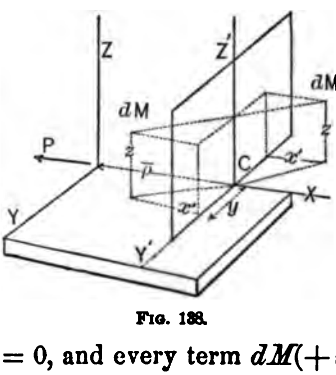


FIG. 138.

Taking the axis X through the centre of gravity, Z being the axis of rotation, Fig. 138, while Z' is the line of symmetry, pass an auxiliary plane $Z'Y'$ parallel to ZY. Then the sum $\int dMxz$ may be written $\int dM(\bar{\rho} + x')z$ which $= \bar{\rho}\int dMz + \int dMx'z$. But $\int dMz = M\bar{z} = 0$, since $\bar{z} = 0$, and every term $dM(+x')z$ is cancelled by a numerically

equal term $dM(-x')z$ of opposite sign. Hence $\int dMxz = 0$. Also $\int dMyz = 0$, since each positive product is annulled by an equal negative one (from symmetry about Z'). Since, also, $\bar{y} = 0$, all six conditions in § 122 are satisfied. Q. E. D.

If the homogeneous body is any solid of revolution *whose geometrical axis is parallel to the axis of rotation*, the foregoing is directly applicable.

122c. *If a homogeneous body revolve uniformly about any axis lying in a plane of symmetry, the acting forces are equivalent to a single force $P = \omega^2 M\bar{\rho}$, acting parallel to the gravity-line which is perpendicular to the axis* (Z), *and away from the centre of gravity, its distance from any origin O in the axis Z being* $=[\int dMxz] \div M\bar{\rho}$ (*the plane ZX being a gravity-plane*).—Fig. 139. From the position of the body we have $\bar{\rho} = \bar{x}$, and $\bar{y} = 0$; hence if a value $\omega^2 M\bar{\rho}$ be given to P and it be made to act through Z and parallel to X, and away from the centre of gravity, all the conditions of § 122 are satisfied except (XII*a.*) and (XIII*a.*). But symmetry about the plane XZ makes $\int dMyz = 0$, and satisfies (XII*a.*), and by placing P at a distance $a = \int dMxz \div M\bar{\rho}$ from O along Z we satisfy (XIII*a.*). Q. E. D.

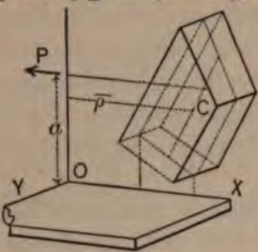


FIG. 139.

Example.—A *slender*, homogeneous, prismatic rod, of length $= l$, is to have a uniform motion, about a vertical axis passing through one extremity, maintained by a cord-connection with a fixed point in this axis. Fig. 140. Given ω, φ, l, ($\bar{\rho} = \frac{1}{2}l \cos \alpha$), and F the cross-section of the rod, let s = the distance from O to any dM of the rod, dM being $= F\gamma ds \div g$. The x of any $dM = s \cos \varphi$; its $z = s . \sin \varphi$;

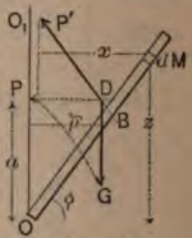


FIG. 140.

$$\therefore \int dMxz = (F\gamma \div g) \sin \varphi \cos \varphi \int_0^l s^2 ds$$

$$= \tfrac{1}{3}(F\gamma l \div g)l^2 \sin \varphi \cos \varphi = \tfrac{1}{3}Ml^2 \sin \varphi \cos \varphi.$$

Hence $a, = \int dMxz \div M\bar{\rho}$, is $= \frac{2}{3}l \sin \varphi$, and the line of action of $P\,(= \omega^2 M\bar{\rho} = \omega^2 (F\gamma l \div g)\,\frac{1}{2}l \cos \varphi)$ is therefore *higher up than the middle of the rod.* Find the intersection D of G and the horizontal drawn through Z at distance a from O. Determine P' by completing the parallelogram GP', attaching the cord so as to make it coincide with P'; for this will satisfy the condition of maintaining the motion, when once begun, viz., that the acting forces G, and the cord-tension P', shall be equivalent to a force $P = \omega^2 M\bar{\rho}$, applied horizontally through Z at a distance a from O.

123. Free Axes. Uniform Rotation.—Referring again to § 122 and Fig. 134, let us inquire under what circumstances the lateral forces, X_1, Y_1, X_0, Y_0, with which the bearings press the axis, to maintain the motion, are individually zero, i.e., *that the bearings are not needed, and may therefore be removed* (except a smooth horizontal plane to sustain the body's weight), leaving the motion undisturbed like that of a top "asleep." For this, not only must ΣX and ΣY both be zero, but also (since otherwise X_1 and X_0 might form a *couple*, or Y_1 and Y_0 similarly) $\Sigma\,(\text{moms.})_X$ and $\Sigma\,(\text{moms.})_Y$ must each $=$ zero. The necessary peculiar distribution of the body's mass about the axis of rotation, then, must be as follows (see the equations of § 122):

First, $\bar{x}$ and $\bar{y}$ each $= 0$, i.e., *the axis must be a gravity-axis.*

Secondly, $\int dMyz = 0$, and $\int dMxz = 0$, the origin being anywhere on Z, the axis of rotation.

An axis (Z) (of a body) fulfilling these conditions is called a **Free Axis**, and since, if either one of the three *Principal Axes* for the centre of gravity (see § 107) be made an axis of rotation (the other two being taken for X and Y), the conditions $\bar{x} = 0$, $\bar{y} = 0$, $\int dMxz = 0$, and $\int dMyz = 0$, are all satisfied, *it follows that every rigid body has at least three free axes, which are the Principal Axes of Inertia of the centre of gravity at right angles to each other.*

In the case of *homogeneous bodies* free axes can often be determined by inspection: e.g., any diameter of a sphere; an

transverse diameter of a right circular cylinder through its centre of gravity, as well as its geometrical axis; the geometrical axis of any solid of revolution; etc.

124. Rotation about an Axis which has a Motion of Translation. —*Take only the particular case where the moving axis is a gravity-axis.* At any instant, let the velocity and acceleration of the axis be v and p; the angular velocity and acceleration about that axis, ω and θ. Then, since the actual motion of a dM in any dt is compounded of its motion of rotation about the gravity-axis and the motion of translation in common with that axis, we may, in forming the imaginary equivalent system in Fig. 141, consider each dM as subjected to the simultaneous action of $dP = dMp$ parallel to X, of the tangential $dT = dM\theta\rho$, and of the normal $dN = dM(\omega\rho)^2 \div \rho = \omega^2 dM\rho$. Take X in the direction of translation, Z (perpendicular to paper through O) is the moving gravity-axis; Y perpendicular to both. At any instant we shall have, then, the following conditions for the acting forces (remembering that $\rho \cos \varphi = y, \int dMy = M\bar{y} = 0$; etc.):

FIG. 141.

$$\Sigma X = \int dP - \int dT \sin \varphi - \int dN \cos \varphi = Mp; \quad . \quad (1)$$

$$\Sigma Y = \int dT \cos \varphi - \int dN \sin \varphi \qquad = 0; \quad . \quad . \quad (2)$$

$$\Sigma \text{ moms.}_Z = \int dT\rho - \int dPy = \theta \int dM\rho^2 = \theta I_Z = \theta M k_Z^2, \quad (3)$$

and three other equations not needed in the following example.

Example.—A homogeneous solid of revolution rolls (without slipping) down a rough inclined plane. Investigate the motion. Considering the body *free*, the acting forces are G (known) and N and P, the unknown normal and tangential components of the action of the plane on the roller. If slipping occurs, then P is the sliding friction due to the pressure N (§ 156); here, however, it is

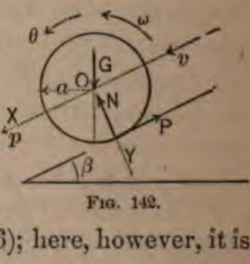


FIG. 142.

less by hypothesis (perfect rolling). At any instant the four unknowns are found by the equations

$$\Sigma X, \text{ i.e., } G \sin \beta - P, = (G \div g)p; \quad . \quad (1)$$

$$\Sigma Y, \text{ i.e., } G \cos \beta - N, = 0; \quad . \quad . \quad . \quad . \quad (2)$$

$$\Sigma \text{ moms.}_Z, \text{ i.e., } \quad Pa, = \theta M k_Z^2; \quad . \quad . \quad (3)$$

while on account of the perfect rolling,

$$\theta a = p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Solving, we have, for the acceleration of translation,

$$p = g \sin \beta \div [1 + (k_Z^2 \div a^2)].$$

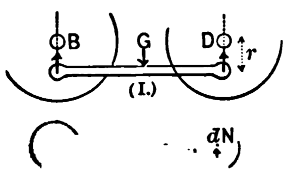
(If the body slid without friction, p would $= g \sin \beta$.) Hence for a cylinder (§ 97), k_Z^2 being $= \frac{1}{2}a^2$, we have $p = \frac{2}{3}g \sin \beta$; and for a sphere (§ 103) $p = \frac{5}{7}g \sin \beta$.

(If the plane is so steep or so smooth that both rolling and slipping occur, then θa no longer $= p$, but the ratio of P to N is known from experiments on sliding friction; hence there are still four equations.)

The motion of translation being thus found to be uniformly accelerated, we may use the equations of § 56 for finding distance, time, etc.

Query.—How may we distinguish two spheres by allowing them to roll down the same inclined plane, if one of them is silver and solid, while the other is of gold, but silvered and hollow, so as to be the same as the first in diameter, weight, and appearance?

125. Parallel-Rod of a Locomotive.—When the locomotive moves uniformly, each dM of the rod between the two (or three) driving-wheels rotates with uniform velocity about a centre of its own on the line BD, Fig. 143, and with a velocity v and radius r common to all, and likewise has a horizontal *uniform* motion of translation. Hence if we inquire what are the reactions P of its supports, as induced *solely* by when in its lowest position (indepe



the rod), we put ΣY of (I.) $= \Sigma Y$ of (II.) (II. shows the imaginary equivalent system), and obtain

$$2P - G = \int dN = \int dMv^2 \div r = (v^2 \div r)\int dM = Mv^2 \div r.$$

Example.—Let the velocity of translation $=$ 50 miles per hour, the radius of the pins be 18 in. $= \frac{3}{2}$ ft., and $=$ *half that of the driving-wheels*, while the weight of the rod is 200 lbs. With $g = 32.2$, we must use the foot and second, and obtain

$$v = \tfrac{1}{2}[50 \times 5280 \div 3600] \text{ ft. per second} = 36.6;$$

while $$M = 200 \div 32.2 = 200 \times .0310 = 6.20;$$

and finally $P = \frac{1}{2}[200 + 6.2(36.6)^2 \div \frac{3}{2}] = 2868.3$ lbs.,

or nearly $1\frac{1}{2}$ tons, *about thirty times that due to the weight alone.*

126. So far in this chapter the motion has been prescribed, and the necessary conditions determined, to be fulfilled by the acting forces at any instant. Problems of a converse nature, i.e., where the initial state of the body and the acting forces are given while the resulting motion is required, are of much greater complexity, but of rare occurrence in practice. The reader is referred to Rankine's Applied Mechanics. A treatment of the Gyroscope will be found in the *American Journal of Science* for 1857, and in the article of that name in Johnson's Cyclopædia.

CHAPTER VI.

WORK, ENERGY, AND POWER.

127. Remark.—These quantities as defined and developed in this chapter, though compounded of the fundamental ideas of matter, force, space, and time, enter into theorems of such wide application and practical use as to more than justify their consideration as separate kinds of quantity.

128. Work in a Uniform Translation. Definition of Work.—Let Fig. 144 represent a rigid body having a motion of translation parallel to X, acted on by a system of forces P_1, P_2, R_3, and R_4, which remain constant.

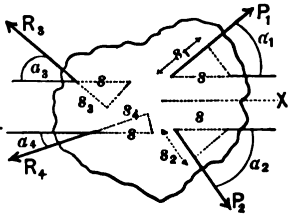


FIG. 144.

Let s be any distance described by the body during its motion; then ΣX must be zero (§ 109), i.e., noting that R_3 and R_4 have negative X components (the supplements of their angles with X are used),

$$P_1 \cos \alpha_1 + P_2 \cos \alpha_2 - R_3 \cos \alpha_3 - R_4 \cos \alpha_4 = 0;$$

or, multiplying by s and transposing, we have (noting that $s_1 \cos \alpha_1 = s_1$ the *projection* of s on P_1, that $s \cos \alpha_2 = s_2$, the *projection* of s on P_2, and so on),

$$P_1 s_1 + P_2 s_2 = R_3 s_3 + R_4 s_4. \quad . \quad . \quad . \quad . \quad (a)$$

The projections s_1, s_2, etc., may be called the *distances described in their respective directions* by the forces P_1, P_2, etc.; P_1 and P_2 having moved *forward*, since s_1 and s_2 fall *in front* of the initial position of their points of application; R_3 and R_4 *backward*, since s_3 and s_4 fall *behind* the initial positions in their case. (By forward and backward we refer to the di-

tion of each force in turn.) The name **Work** is given to the *product of a force by the distance described in the direction of the force by the point of application.* If the force moves *forward* (see above), it is called a *working-force*, and is said *to do* the work (e.g., P_1s_1) expressed by this product; while if *backward*, it is called a *resistance*, and is then said *to have the work* (e.g., R_2s_2), *done upon it*, in *overcoming it* through the distance mentioned (it might also be said to have done negative work).

Eq. (a) above, then, proves the theorem that: *In a uniform translation, the working forces do an amount of work which is entirely applied to overcoming the resistances.*

129. Unit of Work.—Since the work of a force is a product of force by distance, it may logically be expressed as so many foot-pounds, inch-pounds, kilogram-meters, according to the system of units employed. The ordinary English unit is the foot-pound, or ft.-lb. It is of the same quality as a force-moment.

130. Power.—Work as already defined does not depend on the time occupied, i.e., the work P_1s_1 is the same whether performed in a long or short time; but the element of time is of so great importance in all the applications of dynamics, as well as in such practical commercial matters as water-supply, consumption of fuel, fatigue of animals, etc., that the *rate of work* is a consideration both of interest and necessity.

Power is the rate at which work is done, and one of its units is one foot-pound per second in English practice; a larger one will mentioned presently.

The *power exerted* by a *working force*, or *expended* upon a *resistance*, may be expressed symbolically as

$$L = P_1s_1 \div t, \quad \text{or} \quad R_2s_2 \div t,$$

in which t is the time occupied in doing the work P_1s_1 or R_2s_2 (see Fig. 144); or if v_1 is the component in the direction of the force P_1 of the velocity v of the body, we may also write

$$L = P_1v_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

131. *Example.*—Fig. 145, shows as a *free body* a sledge which is being drawn *uniformly* up a rough inclined plane by a cord parallel to the plane. Required the total power exerted (and expended), if the tension in the cord is $P_1 = 100$ lbs., the weight of sledge $R_3 = 160$ lbs., $\beta = 30°$, and the sledge moves 240 ft. each minute. N and R_4 are the normal and parallel (i.e., R_4 = friction) components of the reaction of the plane on the sledge. From eq. (1), § 128, the work done while the sledge advances through $s = 240$ ft. may be obtained either from the working forces, which in this case are represented by P_1 alone, or from the resistances R_3 and R_4. Take the former method first. Projecting s upon P_1 we have $s_1 = s$.

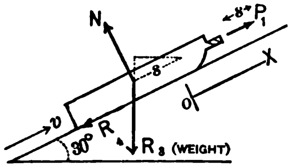


FIG. 145.

Hence P_1s_1 or 100 lbs. $\times$ 240 ft. = 24,000 ft.-lbs.
of work done in 60 seconds. That is, the *power exerted by the working forces* is

$$L = P_1s_1 \div t = 400 \text{ ft.-lbs. per second.}$$

As to the other method, we notice that R_3 and R_4 are resistances, since the projections $s_3 = s \sin \beta$, and $s_4 = s$, would fall back of their points of application in the initial position, while N is *neutral*, i.e., is neither a working force nor a resistance, since the projection of s upon it is zero.

From $\Sigma X = 0$ we have $-R_4 - G \sin \beta + P_1 = 0$,
and from $\Sigma Y = 0$ (§ 109) $N - G \cos \beta = 0$;

whence R_4 the friction = 20 lbs., and N = 138.5 lbs. Also, since $s_3 = s \sin \beta = 240 \times \frac{1}{2} = 120$ ft., and $s_4 = s, = 240$ ft., we have for the work done upon the resistances (i.e., in overcoming them) in 60 seconds

$$R_3s_3 + R_4s_4 = 160 \times 120 + 20 \times 240 = 24,000 \text{ ft.-lbs.,}$$

and the *power expended in overcoming resistances*,

$$L = 24,000 \div 60 = 400 \text{ ft.-lbs. per second,}$$

as already derived. Or, in words the power exerted by the

tension in the co-d is expended entirely in raising the weight a vertical height of 2 feet, and overcoming the friction through a distance of 4 feet along the plane, every second; *the motion being a uniform translation.*

132. Horse-Power.—As an average, a horse can exerts a tractive effort or pull of 100 lbs., at a uniform pace of 4 ft. per second, for ten hours a day without too great fatigue. This gives a power of 400 ft.-lbs. per second; but Boulton & Watt in rating their engines, and experimenting with the strong dray-horses of London, fixed upon 550 ft.-lbs. per second, or 33,000 ft.-lbs. per minute, as a convenient large unit of power. (The French horse-power, or *cheval-vapeur*, is slightly less than the English, being 75 kilogrammeters per second, or 32,550 ft.-lbs. per minute.) This value for the horse-power is in common use. In the example in § 131, then, the power of 400 ft.-lbs. per second exerted in raising the weight and overcoming friction may be expressed as $(400 \div 550 =) \frac{8}{11}$ of a horse-power. A man can work at a rate equal to about $\frac{1}{12}$ of a horse-power, with proper intervals for eating and sleeping.

133. Kinetic Energy. Retarded Translation.—In a retarded translation of a rigid body whose mass $= M$, suppose there are no working-forces, and that the resistances are constant and their resultant is R. (E.g., Fig. 146 shows such a case; a sledge, having an initial velocity c and sliding on a rough horizontal plane, is gradually retarded by the friction R.) R is parallel to the direction of translation (§ 109) and the acceleration is $p = -R \div M$;

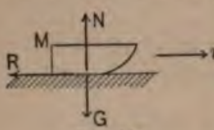


FIG. 146.

hence from $vdv = pds$ we have

$$\int vdv = -(1 \div M)\int Rds. \quad . \quad . \quad . \quad . \quad (1)$$

But the projection of each ds of the motion upon R is $= ds$ itself; i.e. (§ 128), Rds is the *work done upon* R, in overcoming it through the small distance ds, and $\int Rds$ is the sum of all such amounts of work throughout any definite portion of the motion. Let the range of motion be between the points

where the velocity $= c$, and where it $=$ zero (i.e., the mass has come to rest). With these limits in eq. (1) (0 and s' being the corresponding limits for s), we have

$$\frac{Mc^2}{2} = \int_0^{s'} Rds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

That is, ***in giving up all its velocity c the body has been able to do the work*** $\int Rds$ (this, if R remains constant, reduces to Rs') or its equal $\frac{Mc^2}{2}$. If, then, by **energy** we designate the *ability to perform work*, we give the name **kinetic energy** of a moving body to *the product of its mass by half the square of its velocity* $\left(\frac{Mv^2}{2}\right)$; i.e., energy due to motion. (The antiquated term *vis viva* was once applied to the form Mv^2.)

134. Work and Kinetic Energy in any Translation.—Let P be the resultant of the working forces at any instant, R that of the resistances; they (§ 109) will both act in a gravity-line parallel to the direction of translation. The acceleration at any instant is $p = (\Sigma X \div M) = (P - R) \div M$; hence from $vdv = pds$ we have

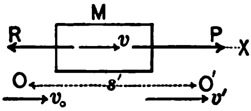


FIG. 147.

$$Mvdv = Pds - Rds. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Integrating between any two points of the motion as O and O' where the velocities are v_0 and v', we have after transposition

$$\int_0^{s'} Pds = \int_0^{s'} Rds + \left[\frac{Mv'^2}{2} - \frac{Mv_0^2}{2}\right]. \quad . \quad . \quad (d)$$

But P being the resultant of P_1, P_2, etc., and R that of R_1, R_2, etc., we may prove, as in § 62, that if du_1, du_2, etc., be the respective projections of any ds upon P_1, P_2, etc., while dw_1, dw_2, etc., are those upon R_1, R_2, etc., then

$$Pds = P_1du_1 + P_2du_2 + \ldots \quad \text{and} \quad Rds = R_1dw_1 + R_2dw_2 \ldots ;$$

and (d) may be rewritten

$$\int_0^{s'} P_1 du_1 + \int_0^{s'} P_2 du_2 + \ldots.$$
$$= \int^{s'} R_1 dw_1 + \int_0^{s'} R_2 dw_2 + \ldots. + \left[\frac{Mv'^2}{2} - \frac{Mv_0^2}{2}\right]; \; (e)$$

or, in words: *In any translation, a portion of the work done by the working forces is applied in overcoming the resistances while the remainder equals the change in the kinetic energy of the body.*

It will be noted that the bracket in (e) depends only on the initial and final velocities, and not upon any intermediate values; hence, if the initial state is one of rest, and also the final, the total change in kinetic energy is zero, and the work of the working forces has been entirely expended in the work of overcoming the resistances; but at intermediate stages the former exceeds the work so far needed to overcome resistances, and this excess is said to be *stored* in the moving mass; and as the velocity gradually becomes zero, this stored energy becomes available for aiding the working forces (which of themselves are then insufficient) in overcoming the resistances, and is then said to be *restored*. (The function of a fly-wheel might be stated in similar terms, but as that involves rotary motion it will be deferred.)

Work applied in increasing the kinetic energy of a body is sometimes called "work of inertia," as also the work done by a moving body in overcoming resistances, and thereby losing speed.

135. Example of Steam-Hammer.—Let us apply eq. (e) to determine the velocity v' attained by a steam-hammer at the lower end of its stroke (the initial velocity being $= 0$), just before delivering its blow upon a forging, supposing that the steam-pressure P_2 at all stages of the downward stroke is given by an *indicator*. Fig. 148. Weight of moving mass is 322 lbs.; $\therefore M = 10$ (foot-pound-second system), $l = 1$ foot. The *working forces* at any instant are $P_1 = G = 322$ lbs.; P_2, which is variable, but whose values at the seven *equally spaced*

points a, b, c, d, e, f, g, are 800, 900, 900, 800, 600, 500, 450 lbs., respectively. R_1 the exhaust-pressure (16 lbs. per sq. inch $\times$ 20 sq. inches piston-area) $=$ 320 lbs., is the only resistance, and is constant. Hence from eq. (e), since here the projections du_1, etc., of any ds upon the respective forces are equal to each other and $= ds$,

$$P_1\int_0^l ds + \int_0^l P_2 ds = R_1\int_0^l ds + \frac{Mv'^2}{2}. \quad (1)$$

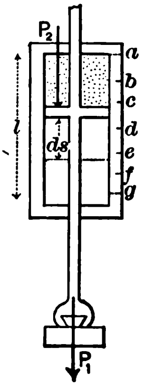


FIG. 148.

The term $\int P_2 ds$ can be obtained approximately by Simpson's Rule, using the above values for six equal divisions, which gives

$$\tfrac{1}{18}[800 + 4(900 + 800 + 500) + 2(900 + 600) + 450]$$

$= 725$ ft.-lbs. of work. Hence, making all the substitutions, we have, since $\int_0^l ds = 1$ ft.,

$$322 \times 1 + 725 = 320 \times 1 + \tfrac{1}{2}Mv'^2; \; \therefore \; \tfrac{1}{2}Mv'^2 = 727 \text{ ft.-lbs.}$$

of energy to be expended in the forging. (Energy is evidently expressed in the same kind of unit as work.) We may then say that the forging receives a blow of 727 ft.-lbs. energy. The pressure actually felt at the surface of the hammer varies from instant to instant during the compression of the forging and the gradual stopping of the hammer, and depends on the readiness with which the hot metal yields.

If the *mean resistance* encountered is R_m, and the depth of compression s'', we would have (neglecting the force of gravity, and noting that now the initial velocity is v', and the final zero), from eq. (c),

$$\tfrac{1}{2}Mv'^2 = R_m s''; \text{ i.e., } R_m = [727 \div s'' \text{ (ft.)}] \text{ lbs.}$$

E.g., if $s'' = \frac{1}{5}$ of an inch $= \frac{1}{60}$ of a foot, $R_m = 43620$ lbs., and the maximum value of R would probably be ... uble this near the end of the impact. If the anvil al... the impact a distance s''', we must substitute ... of s''; this will give a smaller value for ...

By mean value for R is meant [eq. (c)] that value, R_m, which satisfies the relation

$$R_m s' = \int_0^{s'} R ds.$$

This may be called more explicitly a *space-average*, to distinguish it from a *time-average*, which might appear in some problems, viz., a value R_{tm}, to satisfy the relation (t' being the duration of the impact)

$$R_{tm} t' = \int_0^{t'} R dt,$$

and is different from R_m.

From $\frac{1}{2}Mv'^2 = 727$ ft.-lbs., we have $v' = 12.06$ ft. per sec., whereas for a free fall it would have been $\sqrt{2 \times 32.2 \times 1} = 8.03$. (This example is virtually of the same kind as Prob. 4, § 59, differing chiefly in phraseology.)

136. Pile-Driving.—The safe load to be placed upon a pile after the driving is finished is generally taken as a fraction (from $\frac{1}{6}$ to $\frac{1}{8}$) of the resistance of the earth to the passage of the pile as indicated by the effect of the last few blows of the ram, in accordance with the following approximate theory: Toward the end of the driving the resistance R encountered by the pile is nearly constant, and is assumed to be that met by the ram at the head of the pile; the distance s' through which the head of the pile sinks as an effect of the last blow is observed. If G, then, is the weight of the ram, $= Mg$, and h the height of free fall, the velocity due to h, on striking the pile, is $c = \sqrt{2gh}$ (§ 52), and we have, from eq. (c),

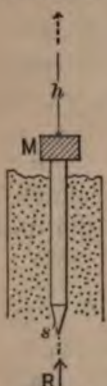


FIG. 149.

$$\frac{1}{2}Mc^2, \text{ i.e., } Gh, = \int_0^{s'} R ds = Rs' \quad . \quad . \quad (1)$$

(R being considered constant); hence $R = Gh \div s'$, and the *safe load* (for ordinary wooden piles),

$$P = \text{from } \tfrac{1}{6} \text{ to } \tfrac{1}{8} \text{ of } Gh \div s'. \quad . \quad . \quad . \quad . \quad (2)$$

Maj. Sanders recommends $\frac{1}{8}$ from experiments made at Fort

Delaware in 1851; Molesworth, $\frac{1}{8}$; General Barnard, $\frac{1}{8}$, from extensive experiments made in Holland.

Of course from eq. (2), given P, we can compute s'.

(Owing to the uncertainty as to how much of the resistance R is due to friction of the soil on the sides of the pile, and how much to the inertia of the soil around the shoe, the more elaborate theories of Weisbach and Rankine seem of little practical account.)

137. *Example.*—In preparing the foundation of a bridge-pier it is found that each pile (placing them 4 ft. apart) must bear safely a load of 12 tons. If the ram weighs one ton, and falls 12 ft., what should be the effect of the last blow on each pile? Using the foot-ton-second system of units, and Molesworth factor $\frac{1}{8}$, eq. (2) gives

$$s' = \tfrac{1}{8}(1 \times 12 \div 72) = \tfrac{1}{48} \text{ of a foot} = \tfrac{1}{4} \text{ of an inch.}$$

That is, the pile should be driven until it sinks only $\frac{1}{4}$ inch under each of the last few blows.

138. Kinetic Energy Lost in Inelastic Direct Central Impact.—Referring to § 60, and using the same notation as there given, we find that if the united kinetic energy possessed by two inelastic bodies after their impact, viz., $\frac{1}{2}M_1C^2 + \frac{1}{2}M_2C^2$, C having the value $(M_1c_1 + M_2c_2) \div (M_1 + M_2)$, be deducted from the amount before impact, viz., $\frac{1}{2}M_1c_1^2 + \frac{1}{2}M_2c_2^2$, the *loss of kinetic energy during impact of two inelastic bodies* is

$$W = \frac{M_1M_2}{M_1 + M_2}(c_1 - c_2)^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

An equal amount of energy is also lost by partially elastic bodies during the first period of the impact, but is partly regained in the second. If the bodies were perfectly elastic, we would find it wholly regained and the resultant loss zero, from the equations of § 60; but this is not quite the reality, on account of internal vibrations.

The *kinetic energy still remaining in two inelastic* l after impact (they move together as one mass) is

$\frac{1}{2}(M_1 + M_2)C^2$, or, after inserting the value of $C = (M_1c_1 + M_2c_2) \div (M_1 + M_2)$, we have

$$W' = \frac{1}{2} \cdot \frac{[M_1c_1 + M_2c_2]^2}{M_1 + M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

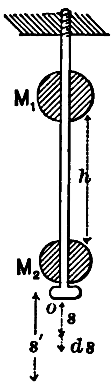


FIG. 150.

Example 1.—The weight $G_1 = M_1g$ falls freely through a height h, impinging upon a weight $G_2 = M_2g$, which was initially at rest. After their (*inelastic*) impact they move on together with the combined kinetic energy just given in (2), which, since c_1 and c_2, the velocities before impact, are respectively $\sqrt{2gh}$ and 0, may be reduced to a simpler form. This energy is soon absorbed in overcoming the flange-pressure R, which is proportional (so long as the elasticity of the rod is not impaired) to the elongation s, as with an ordinary spring. If from previous experiment it is known that a force R_0 produces an elongation s_0, then the variable $R = (R_0 \div s_0)s$. Neglecting the weight of the two bodies as a working force, we now have, from eq. (d),

$$O = \frac{R_0}{s_0}\int_0^{s'} s ds + 0 - \frac{M_1^2gh}{M_1 + M_2};$$

$$\text{i.e.,} \frac{R_0}{s_0} \cdot \frac{s'^2}{2} = \frac{M_1^2gh}{M_1 + M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When $s = s'$, i.e., when the masses are (momentarily) at rest in the lowest position, the flange-pressure or tensile stress in the rod is a maximum, $R' = (R_0 \div s_0)s'$, whence $s' = R's_0 \div R_0$; and (3) may be written

$$\frac{R'}{2}s' = \frac{M_1^2gh}{M_1 + M_2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

or

$$\frac{R'^2s_0}{2R_0} = \frac{M_1^2gh}{M_1 + M_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Eq. (3) gives the final elongation of the rod, and (5) the greatest tensile force upon it, provided the elasticity of the rod is not

impaired. The form $\frac{1}{2}R's'$ in (4) may be looked upon as a direct integration of $\int_0^{s'} R ds$, viz., the mean resistance ($\frac{1}{2}R'$) multiplied by the whole distance (s') gives the work done in overcoming the variable R through the successive ds's.

If the elongation is considerable, the working-forces G_1 and G_2 cannot be neglected, and would appear in the term $+(G_1 + G_2)s'$ in the right-hand members of (3), (4), and (5). The upper end of the rod is firmly fixed, and the rod itself is of small mass compared with M_1 and M_2.

Example 2.—Two cars, Fig. 151, are connected by an elastic chain on a horizontal track. Velocities before impact (i.e., before the stretching of the chain begins, by means of which they are brought to a common velocity at the instant of greatest tension R', and elongation s' of the chain) are $c_1 = c_1$, and $c_2 = 0$.

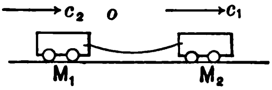


FIG. 151.

During the stretching, i.e., the first period of the impact, the kinetic energy lost by the masses has been expended in stretching the chain, i.e., in doing the work $\frac{1}{2}R's'$; hence we may write (the elasticity of the chain not being impaired) (see eq. (1))

$$\frac{M_1 M_2 c_1^2}{M_1 + M_2} = \frac{1}{2}R's' = \frac{R_0}{s_0} \cdot \frac{s'^2}{2} = \frac{R'^2 s_0}{2R_0}, \quad . \quad . \quad (6)$$

in which the different symbols have the same meaning as in Example 1, in which the rod corresponds to the chain of this example.

(Let the student explain why the stipulation is not made here that one end of the chain shall remain fixed.)

In numerical substitution, 32.2 for g requires the use of the units foot and second for space and time, while the unit of force may be anything convenient.

139. Work and Energy in Rotary Motion. Axis Fixed.—The rigid body being considered free, let an axis through O perpendicular to the paper by the axis of rotation, and resolve all forces not intersecting the axis into components parallel

and perpendicular to the axis, and the latter again into components tangent and normal to the circular path of the point

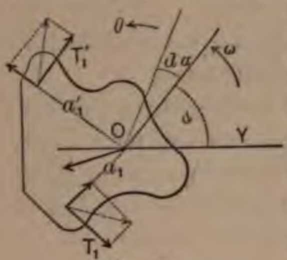


FIG. 152.

of application. These tangential components are evidently the only ones of the three sets mentioned which have moments about the axis, those having moments of the same sign as ω (the angular velocity at any instant) being called *working forces*, T_1, T_2, etc.; those of opposite sign, *resistances*, T_1', T_2', etc.; for when in time dt the point of application B_1, of T_1, describes the small arc $ds_1 = a_1 d\alpha$, whose projection on T_1 is $= ds_1$, this projection falls *ahead* (i.e., in direction of force) of the position of the point at the beginning of dt, while the reverse is true for T_1'.

From eq. (XIV.), § 114, we have for θ (angul. accel.)

$$\theta = \frac{(T_1 a_1 + T_2 a_2 + \ldots.) - (T_1' a_1' + T_2' a_2' + \ldots.)}{I}, \quad (1)$$

which substituted in $\omega d\omega = \theta d\alpha$ (from § 110) gives (remembering that $a_1 d\alpha = ds_1$, etc.), after integration and transposition,

$$\int_0^n T_1 ds_1 + \int_0^n T_2 ds_2 + \text{etc.}$$
$$= \int_0^n T_1' ds_1' + \int_0^n T_2' ds_2' + \text{etc.} + [\tfrac{1}{2}\omega_n^2 I - \tfrac{1}{2}\omega_0^2 I], \quad (2)$$

where 0 and n refer to any two initial and final positions of the rotating body. Eq. (4), § 120, is an example of this.

Now $\frac{1}{2}\omega_n^2 I = \frac{1}{2}\omega_n^2 \int dM\rho^2 = \int \frac{1}{2} dM(\omega_n \rho)^2$, which, since $\omega_n \rho$ is the actual velocity of any dM at this (final) instant, is nothing more than the sum of the amounts of kinetic energy possessed at this instant by all the particles of the body; a similar statement may be made for $\frac{1}{2}\omega_0^2 I$.

Eq. (2) therefore may be put into words as follows:

Between any two positions of a rigid body rotating about a fixed axis, the work done by the working forces is partly used in overcoming the resistances, and the remainder in changing the kinetic energy of the individual particles. If in any case

this remainder is negative, the final kinetic energy is less than the initial, i.e., the work done by the working forces is less than that necessary to overcome the resistances through their respective spaces, and the deficiency is made up by the *restoring* of some of the initial kinetic energy of the rotating body. A moving fly-wheel, then, is a reservoir of kinetic energy.

Eq. (2) has already been illustrated numerically in § 121, where the additional relation was utilized (for a connecting-rod and piston of small mass), that the work done in the steam-cylinder is the same as that done directly at the crank-pin by the working-force there.

140. Work of Equivalent Systems the Same.—*If two plane systems of forces acting on a rigid body are equivalent* (§ 15*a*), *the aggregate work done by either of them during a given slight displacement or motion of the body parallel to their plane is the same.* By aggregate work is meant what has already been defined as the sum of the "virtual moments" (§§ 61 to 64), in any small displacement of the body, viz., the algebraic sum of the products, $\Sigma\,(Pdu)$, obtained by multiplying each force by the projection (du) of the displacement of (or small space described by) its point of application upon the force. (We here class resistances as negative working forces.)

Call the systems A and B; then, if all the forces of B were *reversed in direction* and applied to the body along with those of A, the compound system would be a *balanced system*, and hence we would have (§ 64), for a small motion parallel to the plane of the forces,

$$\Sigma(Pdu) = 0, \text{ i.e., } \Sigma(Pdu) \text{ for } A - \Sigma(Pdu) \text{ for } B = 0,$$

or $$+\,\Sigma(Pdu) \text{ for } A = +\,\Sigma(Pdu) \text{ for } B.$$

But $+\,\Sigma\,(Pdu)$ for A is the aggregate work done by the forces of A during the given motion, and $+\,\Sigma(Pdu)$ for B is a similar quantity for the forces of B (not reversed) during the same small motion if B acted alone. Hence the theorem is proved, and could easily be extended to space of three dimensions.

141. Relation of Work and Kinetic Energy for any Extended Motion of a Rigid Body Parallel to a Plane.—(If at any instant any of the forces acting are not parallel to the plane mentioned, their components lying in or parallel to that plane, will be used instead, since the other components obviously would be neither working forces nor resistances.) Fig. 153 shows an initial position, o, of the body; a final, n; and any intermediate, as q. The forces of the system acting may vary in any manner during the motion.

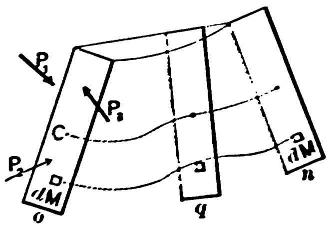


FIG. 153.

In this motion each dM describes a curve of its own with varying velocity v, tangential acceleration p_t, and radius of curvature r; hence in any position q, an imaginary system B (see Fig. 154), equivalent to the actual system A (at q in Fig. 153), would be formed by applying to each dM a tangential force $dT = dMp_t$, and a normal force $dN = dMv^2 \div r$. By an infinite number of consecutive small displacements, the body passes from o to n. In the small displacement of which q is the initial position, each dM describes a space ds, and dT does the work $dTds = dMvdv$, while dN does the work $dN \times 0 = 0$. Hence the total work done by B in the small displacement at q would be

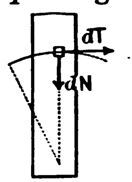


FIG. 154.

$$= dM'v'dv' + dM''v''dv'' + \text{etc.}, \quad . \quad . \quad . \quad (1)$$

including all the dM's of the body and their respective velocities at this instant.

But the work at q in Fig. 153 by the actual forces (i.e., of system A) during the same small displacement must (by § 140) be equal to that done by B, hence

$$P_1du_1 + P_2du_2 + \text{etc.} = dM'v'dv' + dM''v''dv'' + \text{etc.} \quad (q)$$

Now conceive an equation like (q) written out for each of

the small consecutive displacements between positions o and n and corresponding terms to be added; this will give

$$\int_0^n P_1 du_1 + \int_0^n P_2 du_2 + \text{etc.}$$
$$= dM' \int_0^n v'dv' + dM'' \int_0^n v''dv'' + \text{etc.}$$
$$= \tfrac{1}{2}dM'(v_n'^2 - v_0'^2) + \tfrac{1}{2}dM''(v_n''^2 - v_0''^2) + \text{etc.}$$

The second member may be rewritten so as to give, finally,

$$\int_0^n P_1 du_1 + \int_0^n P_2 du_2 + \text{etc.} = \Sigma(\tfrac{1}{2}dMv_n^2) - \Sigma(\tfrac{1}{2}dMv_0^2), \quad \text{(XV.)}$$

or, in words, *the work done by the acting forces* (*treating a resistance as a negative working force*) *between any two positions is equal to the gain* (*or loss*) *in the aggregate kinetic energy of the particles of the body between the two positions.* To avoid confusion, Σ has been used instead of the sign $\int$ in one member of (XV.), in which v_n is the final velocity of any dM (not the same for all necessarily) and v_0 the initial.

(The same method of proof can be extended to three dimensions.)

Since kinetic energy is always essentially positive, if an expression for it comes out negative as the solution of a problem, some impossible conditions have been imposed.

142. Work and Kinetic Energy in a Moving Machine.— Defining a *mechanism* or *machine* as a series of rigid bodies jointed or connected together, so that working-forces applied to one or more may be the means of overcoming resistances occurring anywhere in the system, and also of changing the amount of kinetic energy of the moving masses, let us for simplicity consider a machine the motions of whose parts are all parallel to a plane, and let all the forces acting on any one piece, considered free, at any instant be parallel to the same plane.

Now consider each piece of the machine, or of any series of its pieces, as a free body, and write out eq. (XV.) for it between any two positions (whatever initial and final positions are

selected for the first piece, those of the others must be corresponding initial and corresponding final positions), and it will be found, on adding up corresponding members of these equations, that the terms involving those components of the mutual pressures (between the pieces considered) which are *normal* to the rubbing surfaces at any instant will cancel out, while their components tangential to the rubbing surfaces (i.e., *friction*, since if the surfaces are perfectly smooth there can be no tangential action) will appear in the algebraic addition as resistances multiplied by the distances rubbed through, *measured on the rubbing surfaces.* For example, Fig. 155, where one rotating piece both presses and rubs on another. Let the normal pressure between them at A be $R_2 = P_2$; it is a working force for the body of mass M'', but a resistance for M', hence the separate symbols for the numerically equal forces (action and reaction).

Similarly, the friction at A is $R_3 = P_3$; a resistance for M', a working-force for M''. (In some cases, of course, friction may be a resistance for both bodies.) For a small motion, A describes the small arc AA' about O' in dealing with M', but for M'' it describes the arc AA'' about O'', $A'A''$ being parallel to the surface of contact AD, while AB is perpen-

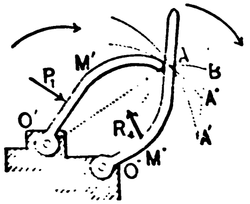


FIG. 155.

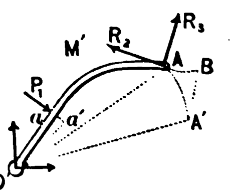


FIG. 156.

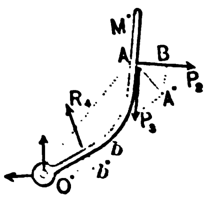


FIG. 157.

dicular to $A'A''$. In Figs. 156 and 157 we see M' and M'' free, and their corresponding small rotations indicated. During these motions the kinetic energy (K. E.) of each mass has changed by amounts $d(\text{K. E.})_{M'}$ and $d(\text{K. E.})_{M''}$ respectively, and hence eq. (XV.) gives, for each free body in turn,

$$P_1\overline{aa'} - R_2\overline{AB} - R_3\overline{A'B} = d(\text{K. E.})_{M'} \quad . \quad (1)$$

$$-R_4\overline{bb''} + P_2\overline{AB} + P_3\overline{A''B} = d(\text{K. E.})_{M''}. \quad . \quad (2)$$

Now add (1) and (2), member to member, remembering that $P_2 = R_2$ and $P_3 = R_3 = F_3 =$ friction, and we have

$$P_1\overline{aa'} - F_3\overline{A'A''} - R_4\overline{bb''} = d(\text{K.E.})_{M'} + d(\text{K.E.})_{M''}, \quad (3)$$

in which the mutual actions of M' and M'' do not appear, except the friction, *the work done in overcoming which, when the two bodies are thus considered collectively, is the product of the friction by the distance $A'A''$ of actual rubbing measured on the rubbing surface.* For any number of pieces, then, *considered free collectively*, the assertion made at the beginning of this article is true, since any finite motion consists of an infinite number of small motions to each one of which an equation like (3) is applicable.

Summing the corresponding terms of all such equations, we have

$$\int_0^n P_1du_1 + \int_0^n P_2du_2 + \text{etc.} = \Sigma(\text{K.E.})_n - \Sigma(\text{K.E.})_0. \text{(XVI.)}$$

This is of the same form as (XV.), but instead of applying to a single rigid body, deals with any assemblage of rigid parts forming a machine, or any part of a machine (a similar proof will apply to three dimensions of space); but it must be remembered that it excludes all the *mutual* actions of the pieces considered except friction, which is to be introduced in the manner just illustrated. A flexible inextensible cord may be considered as made up of a great number of short rigid bodies jointed without friction, and hence may form part of a machine without vitiating the truth of (XVI.).

$\Sigma(\text{K. E.})_n$ signifies the sum obtained by adding the amounts of kinetic energy ($\frac{1}{2}dMv_n^2$ for each elementary mass) possessed by all the particles of all the rigid bodies at their final positions; $\Sigma(\text{K. E.})_0$, a similar sum at their initial positions. For example, the K. E. of a rigid body having a motion of translation of velocity v, $= \frac{1}{2}v^2\int dM = \frac{1}{2}Mv^2$; that of a rigid body having an angular velocity ω about a fixed axis Z, $= \frac{1}{2}\omega^2 I_Z$ (§ 139); while, if it has an angular velocity ω about a gravity-

axis Z, which has a velocity v_Z of translation at right angles to itself, the (K. E.) at this instant may be proved to be

$$\tfrac{1}{2}Mv_Z^2 + \tfrac{1}{2}\omega^2 I_Z,$$

i.e., is the sum of the amounts *due to the two motions separately*.

143. K. E. of Combined Rotation and Translation.—The last statement may be thus proved. Fig. 157. At a given instant the velocity of any dM is v, the diagonal formed on the velocity v_Z of translation, and the rotary velocity $\omega\rho$ relatively to the moving gravity-axis Z (perpendicular to paper) (see § 71),

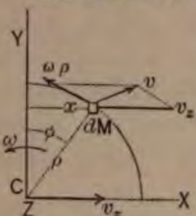


FIG. 158.

i.e., $v^2 = v_Z^2 + (\omega\rho)^2 + 2(\omega\rho)v_Z \cos\varphi$;

hence we have K. E., at this instant,

$$= \int \tfrac{1}{2}dMv^2 = \tfrac{1}{2}v_Z^2\int dM + \tfrac{1}{2}\omega^2\int dM\rho^2 + 2\omega v_Z\int dM\rho\cos\varphi,$$

but $\rho\cos\varphi = x$, and $\int dMx = M\bar{x} = 0$, since Z is a gravity-axis,

$$\therefore \text{K. E.} = \tfrac{1}{2}Mv_Z^2 + \tfrac{1}{2}\omega^2 I_Z. \quad \text{Q. E. D.}$$

It is interesting to notice that the K. E. due to rotation, viz., $\tfrac{1}{2}\omega^2 I_Z = \tfrac{1}{2}M(\omega k)^2$, is the same as if the whole mass were concentrated in a point, line, or thin shell, at a distance k, the radius of gyration, from the axis.

144. Example of a Machine in Operation.—Fig. 159. Consider the four consecutive moving masses, M', M'', M''', and M^{iv} (being the piston; connecting-rod; fly-wheel, crank, drum, and chain; and weight on inclined plane) as *free*, collectively. Let us apply eq. (XVI.), the initial and final positions being taken when the crank-pin is at its dead-points o and n; i.e., we deal with the progress of the pieces made while the crank-pin describes its upper semicircle. Remembering that the mutual actions between any two of these four masses can be left out of account (except friction), the only forces to be put in are the actions of other bodies on any one of these four, and are

shown in the figure. The only *mutual* friction considered will be at the crank-pin, and if this as an average $= F'$, the work done on it between o and $n = F'\pi r''$, where $r'' =$ radius of crank-pin. The work done by P_1 the effective steam-pressure (let it be constant) during this period is $= P_1 l'$; that done in overcoming F_1, the friction between piston and cylinder, $= F_1 l'$; that done *upon* the weight G' of connecting-rod is cancelled by the work done *by it* in the descent following; the work done

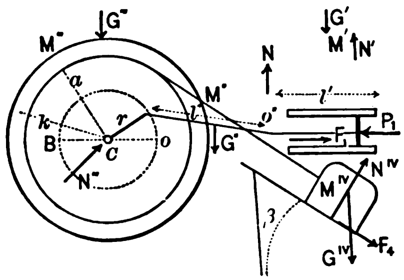


FIG. 159.

upon G^{iv}, $= G^{\text{iv}}\pi a \sin \beta$, where $a =$ radius of drum; that upon the friction F_4, $= F_4\pi a$. The pressures N', N^{iv}, and N''', and weights G' and G''', are neutral, i.e., do no work either positive or negative. Hence the left-hand member of (XVI.) becomes, between o and n,

$$P_1 l' - F_1 l' - F'\pi r - G^{\text{iv}}\pi a \sin \beta - F_4\pi a, \quad . \quad . \quad (1)$$

provided the respective distances are *actually described* by these forces, i.e., if the masses have sufficient initial kinetic energy to carry the crank-pin beyond the point of minimum velocity, with the aid of the working force P_1, whose effect is small up to that instant.

As for the total initial kinetic energy, i.e., $\Sigma(\text{K. E.})_0$, let us express it in terms of the velocity of crank-pin at o, viz., V_0. The $(\text{K.E.})_0$ of M' is nothing; that of M'', which at this instant is rotating about its right extremity (*fixed* for the instant) with angular velocity $\omega'' = V_0 \div l''$, is $\frac{1}{2}\omega''^2 I_{O''}''$; that of M''' $= \frac{1}{2}\omega'''^2 I_C'''$, in which $\omega''' = V_0 \div r$; that of M^{iv} (translation) $= \frac{1}{2}M^{\text{iv}}v_0^{\text{iv}2}$, in which $v_0^{\text{iv}} = (a \div r)\, V_0$. $\Sigma(\text{K. E.})_n$ is expressed

in a corresponding manner with V_n (final velocity of crank-pin) instead of V_0. Hence the right-hand member of (XVI.) will give (putting the radius of gyration of M'' about $O'' = k''$, and that of M''' about $C = k$)

$$\tfrac{1}{2}(V_n^2 - V_0^2)\left[M''\frac{k''^2}{l''^2} + M'''\frac{k^2}{r^2} + M^{\text{iv}}\frac{a^2}{r^2}\right]. \quad . \quad . \quad (2)$$

By writing (1)=(2), we have an equation of condition, capable of solution for any one unknown quantity, to be satisfied for the extent of motion considered. It is understood that the chain is always taut, and that its weight and mass are neglected.

145. Numerical Case of the Foregoing.—(Foot-pound-second system of units for space, force, and time; this requires $g = 32.2$.)

Suppose the following data:

FEET.	LBS.	LBS.	MASS UNITS.
			(and ∴)
$l' = 2.0$	$P_1 = 6000$	$G' = 60$	$M' = 1.86$
$l'' = 4.0$	$F_1 = 100$	$G'' = 50$	$M'' = 1.55$
$a = 1.5$	F'' (av'ge) $= 40$	$G''' = 400$	$M''' = 12.4$
$r = 1.0$	$F_4 = 300$	$G^{\text{iv}} = 3220$	$M^{\text{iv}} = 100.0$
$k = 1.8$		Also let $V_0 = 4$ ft. per sec.	
$k'' = 2.3$			

Denote (1) by W and the large bracket in (2) by $\overline{M}$ (this by some is called the total mass "*reduced*" to the crank-pin). Putting (1) = (2) we have, solving for the unknown V_n,

$$V_n = \sqrt{\frac{2\,W}{\overline{M}} + V_0^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For above values,

$$W = 12{,}000 - 400 - 125.7 - 7590.0 - 1417.3$$
$$= 2467 \text{ foot-pounds};$$

while $\overline{M} = 0.5 + 40.3 + 225.0 = 265.8$ mass-units;

whence

$$V_n = \sqrt{18.56 + 16} = \sqrt{34.56} = 5.88 \text{ ft. per second.}$$

As to whether the crank-pin actually reaches the dead-point B, requires separate investigations to see whether V becomes zero or negative between o and B (a negative value is inadmissible, since a reversal of direction implies a different value for W), i.e., whether the proposed extent of motion is realized; and these are made by assigning some other intermediate position m, as a final one, and computing V_m, remembering that when m is not a dead-point the $(K.E.)_m$ of M' is not zero, and must be expressed in terms of V_m, and that the $(K.E.)_m$ of the connecting-rod M'' must be obtained from § 143.

146. Regulation of Machines.—As already illustrated in several examples (§ 121), a fly-wheel of sufficient weight and radius may prevent too great fluctuation of speed in a single stroke of an engine; but to prevent a permanent change, which must occur if the work of the working force or forces (such as the steam-pressure on a piston, or water-impulse in a turbine) exceeds for several successive strokes or revolutions the work required to overcome resistances (such as friction, gravity, resistance at the teeth of saws, etc., etc.) through their respective spaces, automatic governors are employed to diminish the working force, or the distance through which it acts per stroke, until the normal speed is restored; or *vice versâ*, if the speed slackens, as when new resistances are temporarily brought into play. Hence when several successive periods, strokes (or other cycle), are considered, the kinetic energy of the moving parts will disappear from eq. (XVI.), leaving it in this form:

work of working-forces = work done upon resistances.

147. Power of Motors.—In a mill where the same number of machines are run continuously at a constant speed proper for their work, turning out per hour the same number of barrels of flour, feet of lumber, or other commodity, the motor (e.g., a steam-engine, or turbine) works at a constant rate, i.e., develops a definite horse-power (H.P.), which is thus found in the case of *steam-engines* (double-acting):

$$\text{H.P.} = \left\{\begin{array}{l}\text{total mean effective}\\ \text{steam-pressure on}\\ \text{piston in lbs.}\end{array}\right\} \times \left\{\begin{array}{l}\text{distance in feet}\\ \text{travelled by pis-}\\ \text{ton per second.}\end{array}\right\} \div 550,$$

i.e., the work (in ft.-lbs) done per second by the working force divided by 550 (see § 132). The total effective pressure at any instant is the excess of the forward over the back-pressure, and by its mean value (since steam is usually used expansively) is meant such a value P' as, multiplied by the length of stroke l, shall give

$$P'l = \int_0^l Pdx,$$

where P is the variable effective pressure and dx an element of its path. If u is the number of strokes per second, we may also write (*foot-pound-second system*)

$$\text{H.P.} = P'lu \div 550 = \left[\int_0^l Pdx\right]u \div 550. \quad \text{(XVII.)}$$

Very often the number of revolutions *per minute*, m, of the crank is given, and then

$$\text{H.P.} = P' \text{(lbs.)} \times 2l \text{(feet)} \times m \div 33{,}000.$$

If F = area of piston we may also write $P' = Fp'$, where p' is the mean effective steam-pressure per unit of area. Evidently, to obtain P' in lbs., we multiply F in sq. in. by p' in lbs. per sq. in., or F in sq. ft. by p' in lbs. per sq. foot; the former is customary. p' in practice is obtained by measurements and computations from "indicator-cards" (see § 135, in which $(P_2 - R_1)$ corresponds to P of this section); or $P'l$, i.e., $\int_0^l Pdx$, may be computed theoretically as in § 59, Problem 4.

The power as thus found is expended in overcoming the friction of all moving parts (which is sometimes a large item), and the resistances peculiar to the kind of work done by the machines. The work periodically *stored* in the increased kinetic energy of the moving masses is *restored* as they periodically resume their minimum velocities.

148. Potential Energy.—There are other ways in which work or energy is stored and then restored, as follows:

First. In raising a weight G through a height h, an amount of work $= Gh$ is done *upon* G, as a *resistance*, and if at any subsequent time the weight is allowed to descend through the same vertical distance h (the form of path is of no account), G, now a *working force*, *does* the work Gh, and thus in aiding the motor repays, or restores, the Gh expended by the motor in raising it. If h is the vertical height through which the centre of gravity rises and sinks periodically in the motion of the machine, the force G may be left out of account in reckoning the expenditure of the motor's work, and the body when at its highest point is said to possess an amount Gh of **potential energy,** i.e., *energy of position*, since it is capable of doing the work Gh in sinking through its vertical range of motion.

Second. So far, all bodies considered have been by express stipulation *rigid*, i.e., incapable of changing shape. To see the effect of a lack of rigidity as affecting the principle of work and energy in machines, take the simple case in Fig. 160. A helical spring at a given instant is acted on at each end by a force P in an axial direction (they are equal, supposing the mass of the spring small).

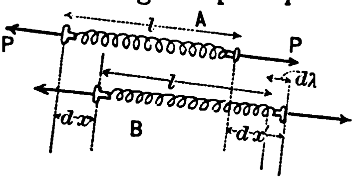


FIG. 160.

As the machine operates of which it is a member, it moves to a new consecutive position B, suffering a further elongation $d\lambda$ in its length (if P is increasing). P on the right, a working force, does the work Pdx'; how is this expended? P on the left has the work Pdx done upon it, and the mass is too small to absorb kinetic energy or to bring its weight into consideration. The remainder, $Pdx' - Pdx = Pd\lambda$, is expended in stretching the spring an additional amount $d\lambda$, and is capable of restoration if the spring retains its elasticity. Hence the work done in changing the form of bodies *if they are elastic* is said to be stored in the form of **potential energy.** That is, in the operation of machines, the name *potential energy* is also given to the energy

stored and restored periodically in the changing and regaining of form of elastic bodies.

149. Other Forms of Energy.—Numerous experiments with various kinds of apparatus have proved that for every 772 (about) ft.-lbs. of work spent in overcoming friction, one British unit of heat is produced (viz., the quantity of heat necessary to raise the temperature of one pound of water from 32° to 33° Fahrenheit); while from converse experiments, in which the amount of heat used in operating a steam-engine was all carefully estimated, the disappearance of a certain portion of it could only be accounted for by assuming that it had been converted into work at the same rate of (about) 772 ft.-lbs. of work to each unit of heat (or 425 kilogrammetres to each French unit of heat). This number 772, or 425, according to the system of units employed, is called the *Mechanical Equivalent of Heat*, first discovered by Joule and confirmed by Hirn.

Heat then is energy, and is supposed to be of the kinetic form due to the rapid motion or vibration of the molecules of a substance. A similar agitation among the molecules of the (hypothetical) ether diffused through space is supposed to produce the phenomena of light, electricity, and magnetism. Chemical action being also considered a method of transforming energy (its possible future occurrence as in the case of coal and oxygen being called potential energy), the well-known doctrine of the *Conservation of Energy*, in accordance with which energy is indestructible, and the doing of work is simply the conversion of one or more kinds of energy into equivalent amounts of others, is now one of the accepted hypotheses of physics.

Work consumed in friction, though practically lost, still remains in the universe as heat, electricity, or some other subtile form of energy.

150. Power Required for Individual Machines. Dynamometers of Transmission.—If a machine is driven by an endless belt from the main-shaft, A, Fig. 161, being the driving-pulley

on the machine, the working force which drives the machine, in other words the "grip" with which the belt takes hold of the pulley tangentially, $= P - P'$, P and P' being the tensions in the "driving" and "following" sides of the belt respectively. The belt is supposed not to slip on the pulley. If v is the velocity of the pulley-circumference, the work expended on the machine per second, i.e., the *power*, is

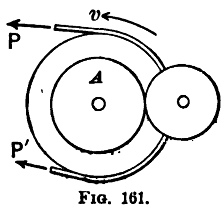


FIG. 161.

$$L = (P - P')v. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To measure the force $(P - P')$, an apparatus called a *Dynamometer of Transmission* may be placed between the main shaft and the machine, and the belt made to pass through it in such a way as to measure the tensions P and P', or principally their difference, without meeting any resistance in so doing; that is, the power is *transmitted*, not absorbed, by the apparatus. One invention for this purpose (mentioned in the *Journal of the Franklin Institute* some years ago) is shown (*in principle*) in Fig. 162. A vertical plate carrying four pulleys and a scale-pan is first balanced on the pivot C. The belt being then adjusted, as shown, and the power turned on, a sufficient weight G is placed in the scale-pan to balance the plate again, for whose equilibrium we must have $Gb = Pa - P'a$, since the P and P' on the right are purposely given no leverage about C. The velocity of belt, v, is obtained by a simple counting device. Hence $(P - P')$ and v become known, and $\therefore$ L from (1).

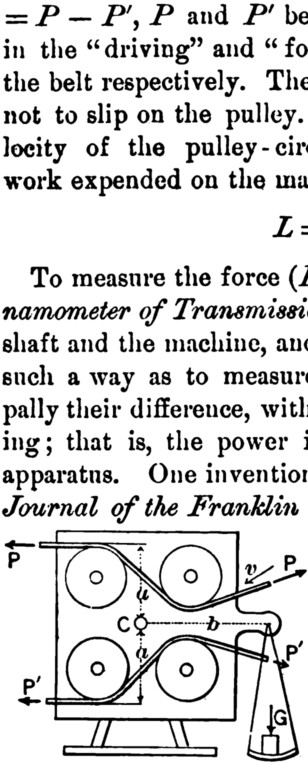


FIG. 162.

Many other forms of transmission-dynamometers are in use, some applicable whether the machine is driven by belting or gearing from the main shaft. Emerson's *Hydrodynamics* describes his own invention on p. 283, and gives results of measurements with it; e.g., at Lowell, Mass., the power required to drive 112 looms, weaving 36-inch sheetings, No. 20 yarn,

60 threads to the inch, speed 130 picks to the minute, was found to be 16 H.P., i.e., $\frac{1}{4}$ H.P. to each loom (p. 335).

151. Dynamometers of Absorption.—These are so named since they furnish in themselves the resistance (friction or a weight) in the overcoming (or raising) of which the power is expended or absorbed. Of these the ***Prony Friction Brake*** is the most common, and is used for measuring the power developed by a given motor (e.g., a steam-engine or turbine) not absorbed in the friction of the motor itself. Fig. 163

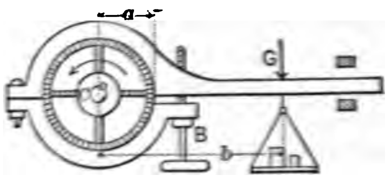


FIG. 163.

shows one fitted to a vertical pulley driven by the motor. By tightening the bolt B, the velocity v of pulley-rim may be made constant at any desired value (within certain limits) by the consequent friction. v is measured by a counting apparatus, while the friction (or *tangential* components of action between pulley and brake), $= F$, becomes known by noting the weight G which must be placed in the scale pan to balance the arm between the checks; then

$$Fa = Gb, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for the equilibrium of the brake (supposing the weight of brake and scale-pan previously balanced on C) and the work done per unit of time, or *power*, is

$$L = Fv. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

A "dash-pot" is frequently connected with the arm to prevent sudden oscillations. In case the pulley is horizontal, a bell-crank lever is added between the arm and the scale-pan, and then eq. (1) will contain two additional lever-arms.

152. The Indicator, used with steam and other fluid engines, is a special kind of dynamometer in which the automatic motion of a pencil describes a curve on paper whose ordinates are proportional to the fluid pressures exerted in the cylinder at successive points of the stroke. Thus, Fig. 164, the back-pressure being constant and $= P_b$,

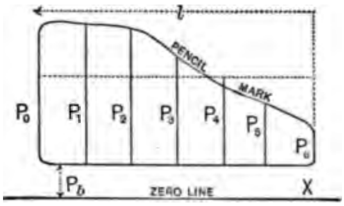


FIG. 164.

the ordinates P_0, P_1, etc., represent the effective pressures at equally spaced points of division. The mean effective pressure P' (see § 147) is, for this figure, by Simpson's Rule (six equal spaces),

$$P' = \tfrac{1}{18}[P_0 + 4(P_1 + P_3 + P_5) + 2(P_2 + P_4) + P_6].$$

This gives a near approximation. The power is now found by § 147.

153. The theory of **Atwood's Machine** is most directly expressed by the principle of work and energy; i.e., by eq. (XVI.), § 142. Fig. 165. The parts considered free, collectively, are the rigid bodies P, Q, G, and four friction-wheels like G_1; and the flexible cord, which does not slip on the upper pulley. There is no slipping at D, hence no sliding friction there. The actions of external bodies on these eight consist of the working force P, the resistances Q and the four F's (at bearings of friction-wheel axles); all others (G, $4G_1$, and the four R's) are neutral. Since there is no rubbing between any two of the eight bodies, no mutual actions whatever will enter the equation. Let $P > Q$, and I and I_1 be the moments of inertia of G and G_1, respectively, about their respective axes of figure. Let the apparatus start from rest, then when P has descended through any vertical distance s, and ac-

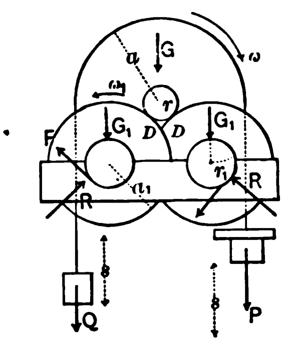


FIG. 165.

quired the velocity v, Q has been drawn up an equal distance and acquired the same velocity, while the pulley G has acquired an angular velocity $\omega = v \div a$, each friction-pulley an angular velocity $\omega_1 = (r : a)v \div a_1$. As to the forces, P has done the work Ps, Q has had the work Qs done upon it, while each F has been overcome through the space $(r_1 : a_1)(r : a)s$; all the other forces are neutral. Hence, from eq. (XVI.), § 142 (see also § 139), we have

$$Ps - Qs - 4F\frac{r_1}{a_1} \cdot \frac{r}{a}s$$
$$= \left[\frac{P}{g} + \frac{Q}{g}\right]\frac{v^2}{2} + \frac{1}{2}\,\frac{v^2}{a^2}I + \frac{1}{2} \cdot \frac{r^2}{a^2} \cdot \frac{v^2}{a_1^{\,2}}4I_1 - 0.$$

Evidently $v = \sqrt{s} \times$ constant, i.e., the motion of P and Q is uniformly accelerated. If, after the observed space s has been described, P is suddenly diminished to such a value P' that the motion continues with a constant velocity $= v$, we shall have, for any further space s',

$$P's' - Qs' - 4F\frac{r_1}{a_1} \cdot \frac{r}{a}s' = 0,$$

from which F can be obtained (nearly); while if t' be the observed time of describing s', $v = s' \div t'$ becomes known. Also we may write $I = (G \div g)k^2$ and $I_1 = (G_1 \div g)k_1^{\,2}$, and thus finally compute the acceleration of gravity, g, from our first equation above.

154. Boat-Rowing.—Fig. 166. During the stroke proper, let P = mean pressure on one oar-handle; hence the pressures on the foot-rest are $2P$, resistances. Let M = mass of boat and load, v_0 and v_n its velocities at beginning and end of stroke. P_1 = pressures between oar-blade and water. R = mean resistance of water to the boat's passage at this (mean) speed. These are the only (horizontal) forces to be considered as acting on the boat and two oars, considered free collectively. During the stroke the boat describes the space $s_3 = CD$, the oar-handle the space $s_2 = AB$, while the oar-blade slips back-

ward through the small space (the "slip") $= s_1$ (average). Hence by eq. (XVI.), § 142,

$$2Ps_2 - 2Ps_1 - Rs_2 - 2P_1s_1 = \frac{1}{2}M(v_n^2 - v_0^2);$$

i.e., $2P(s_2 - s_1) = 2P \times \overline{AE} = 2Ps = Rs_2 + 2P_1s_1 + \frac{1}{2}M(v_n^2 - v_0^2)$;

or, in words, the product of the oar-handle pressures into the distance described by them *measured on the boat*, i.e., the work done by these pressures *relatively to the boat*, is entirely accounted for in the work of slip and of liquid-resistance, and in-

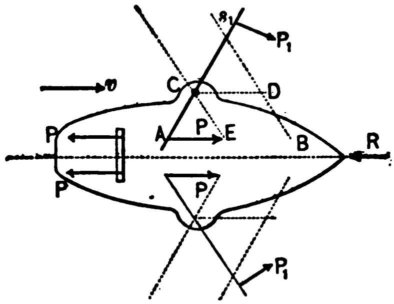


FIG. 166.

creasing the kinetic energy of the mass. (The useless work due to slip is inevitable in all paddle or screw propulsion, as well as a certain amount lost in machine-friction, not considered in the present problem.) During the "recover" the velocity decreases again to v_0.

155. Examples.—1. What work is done on a level track, in bringing up the velocity of a train weighing 200 tons, from zero to 30 miles per hour, if the total frictional resistance (at any velocity, say) is 10 lbs. per ton, and if the change of speed is accomplished in a length of 3000 feet?

(*Foot-ton-second system.*) 30 miles per hour = 44 ft. per sec. The mass

$$= 200 \div 32.2 = 6.2;$$

∴ the change in kinetic energy,

$$(= \tfrac{1}{2}Mv^2 - \tfrac{1}{2}M \times O^2),$$
$$= \tfrac{1}{2}(6.2) \times 44^2 = 6001.6 \text{ ft.-tons.}$$

The work done in overcoming friction $= Fs$, i.e.,

$= 200 \times 10 \times 3000 = 6{,}000{,}000$ ft.-lbs. $= 3000$ ft.-tons;

$\therefore$ total work $= 6001.6 + 3000 = 9001.6$ ft.-tons.

(If the track were an up-grade, 1 in 100 say, the item of $200 \times 30 = 6000$ ft.-tons would be added.)

Example 2.—Required the rate of work, or power, in Example 1. The power is variable, depending on the velocity of the train at any instant. Assume the motion to be uniformly accelerated, then the working force is constant; call it P. The acceleration (§ 56) will be $p = v^2 \div 2s = 1936 \div 6000 = 0.322$ ft. per sq. sec.; and since $P - F = Mp$, we have

$$P = 1 \text{ ton} + (200 \div 32.2) \times 0.322 = 3 \text{ tons},$$

which is $6000 \div 200 = 30$ lbs. per ton of train, of which 20 is due to its inertia, since when the speed becomes uniform the work of the engine is expended on friction alone.

Hence when the velocity is 44 ft. per sec., the engine is working at the rate of $Pv = 264{,}000$ ft.-lbs. per sec., i.e., at the rate of 480 H. P.;

At $\frac{1}{4}$ of 3000 ft. from the start, at the rate of 240 H. P., half as much;

At a uniform speed of 30 miles an hour the power would be simply $1 \times 44 = 44$ ft.-tons per sec. $= 160$ H. P.

Example 3.—The resistance offered by still water to the passage of a certain steamer at 10 knots an hour is 15,000 lbs. What power must be developed by its engines, at this uniform speed, considering no loss in "slip" nor in friction of machinery?

Example 4.—Same as 3, except that the speed is to be 15 knots (i.e., nautical miles; each $= 6086$ feet) an hour, assuming that the resistances are as the square of the speed (approximately true).

Example 5.—Same as 3, except that 12% of the power is absorbed in the "slip" (i.e., in pushing aside and backwards the water acted on by the screw or paddle), and 8% in friction of machinery.

Example 6.—In Example 3, if the crank-shaft makes 60

revolutions per minute, the crank-pin describing a circle of 18 inches radius, required the average value of the tangential component of the thrust (or pull) of the connecting-rod against the crank-pin.

Example 7.—A solid sphere of cast-iron is *rolling* up an incline of 30°, and at a certain instant its centre has a velocity of 36 inches per second. Neglecting friction of all kinds, how much further will the ball mount the incline (see § 143)?

Example 8.—In Fig. 163, with $b = 4$ ft. and $a = 16$ inches, it is found in one experiment that the friction which keeps the speed of the pulley at 120 revolutions per minute is balanced by a weight $G = 160$ lbs. Required the power thus measured.

Although in Examples 1 to 6 the steam cylinder is itself in motion, the work per stroke is still = mean effective steam-pressure on piston × length of stroke, for this is the final form to which the separate amounts of work done by, or upon, the two cylinder heads and the two sides of the piston will reduce, when added algebraically. See § 154.

CHAPTER VII.

FRICTION.

156. Sliding Friction.—When the surfaces of contact of two bodies are perfectly smooth, the direction of the pressure or pair of forces between them is normal to these surfaces, i.e., to their tangent-plane; but when they are rough, and moving one on the other, the forces or actions between them incline away from the normal, each on the side opposite to the direction of the (relative) motion of the body on which it acts. Thus, Fig. 167, a block whose weight is G, is drawn on a rough horizontal table by a horizontal cord, the tension in which is P. On account of the roughness of one or both bodies the action of the table upon the block is a force P_1, inclined to the normal (which is vertical in this case) at an angle $= \varphi$ *away* from the direction of the relative velocity v. This angle φ is called the *angle of friction*, while the tangential component of P_1 is called the *friction* $= F$. The normal component N, which in this case is equal and opposite to G the weight of the body, is called the *normal pressure.*

FIG. 167.

Obviously $F = N \tan \varphi$, and denoting $\tan \varphi$ by f, we have

$$F = fN. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

f is called the *coefficient of friction*, and may also be defined as the ratio of the friction F to the normal pressure N which produces it.

In Fig. 167, if the motion is accelerated (acc. $=p$), we have (eq. (IV.), § 55) $P - F = Mp$; if uniform, $P - F = 0$; from which equations (see also (1)) f may be computed. In the latter case f may be found to be different with different velocities (the surfaces retaining the same character of course), and then a uniformly accelerated motion is impossible unless P were to vary as F.

As for the lower block or table, forces the equals and opposites of N and F (or a single force equal and opposite to P_1) are comprised in the system of forces acting upon it.

As to whether F is a *working force* or a *resistance*, when either of the two bodies is considered free, depends on the circumstances of its motion. For example, in friction-gearing the tangential action between the two pulleys is a resistance for one, a working force for the other.

If the force P, Fig. 167, is just sufficient to start the body, or is just on the point of starting it (this will be called *impending motion*), F is called the *friction of rest.* If the body is at rest and P is not sufficient to start it, the tangential component will then be $<$ the friction of rest, viz., just $= P$. As P increases, this component continually equals it in value, and P_1 acquires a direction more and more inclined from the normal, until the instant of impending motion, when the tangential component $= fN =$ the *friction of rest.* When motion is once in progress, the friction, called then the *friction of motion*, $= fN$, in which f is not necessarily the same as in the friction of rest.

157. Laws of Sliding Friction.—Experiment has demonstrated the following relations approximately, for two given rubbing surfaces:

(1) The coefficient, f, is independent of the normal pressure N.

(2) The coefficient, f, for friction of motion, is the same at all velocities.

(3) The coefficient, f, for friction of rest (i.e., impending motion) is usually greater than that for friction of motion (probably on account of adhesion).

(4) The coefficient, f, is independent of the extent of rubbing surface.

(5) The interposition of an unguent (such as oil, lard, tallow, etc.) diminishes the friction very considerably.

158. Experiments of Sliding Friction.—These may be made with simple apparatus. If a block of weight $= G$, Fig. 168, be placed on an inclined plane of uniformly rough surface, and the latter be gradually more and more inclined from the horizontal until the block *begins* to move, the value of β at

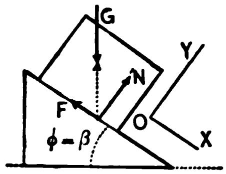


FIG. 168.

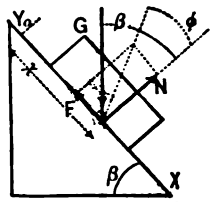


FIG. 169.

this instant $= \varphi$, and $\tan \varphi = f =$ coefficient of friction of rest. For from $\Sigma X = 0$ we have F, i.e., fN, $= G \sin \beta$; from $\Sigma Y = 0$, $N = G \cos \beta$; whence $\tan \beta = f$, $\therefore \beta$ must $= \varphi$.

Suppose β so great that the motion is accelerated, the body starting from rest at o, Fig. 169. It will be found that the distance x varies as the square of the time, hence (§ 56) the motion is uniformly accelerated (along the axis X). (Notice in the figure that G is no longer equal and opposite to P_{11}, the resultant of N and F, as in Fig. 168.)

$$\Sigma Y = 0, \quad \text{which gives } N - G \cos \beta = 0;$$

$$\Sigma X = Mp, \text{ which gives } G \sin \alpha - fN = (G \div g)p;$$

while (from § 56) $\qquad p = 2x \div t^2.$

Hence, by elimination, x and the corresponding time t having been observed, we have for the coefficient of friction of motion

$$f = \tan \beta - \frac{2x}{gt^2 \cos \beta}.$$

In view of (3), § 157, it is evident that if a value β_m has been found experimentally for β such that the block, *once started by hand*, preserves a uniform motion down the plane, then, since $\tan \beta_m = f$ for friction of motion, β_m may be less than the β in Fig. 168, for friction of rest.

159. Another apparatus consists of a horizontal plane, a pulley, cord, and two weights, as shown in Fig. 59. The masses of the cord and pulley being small and hence neglected, the analysis of the problem when G is so large as to cause an accelerated motion is the same as in that example [(2) in § 57], except in Fig. 60, where the frictional resistance fN should be put in pointing toward the left. N still $= G_1$, and $\therefore$

$$S - fG_1 = (G_1 \div g)p; \quad . \quad . \quad . \quad . \quad . \quad (1)$$

while for the other free body in Fig. 61 we have, as before,

$$G - S = (G \div g)p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) and (2), S the cord-tension can be eliminated, and solving for p, writing it equal to $2s \div t^2$, s and t being the observed distance described (from rest) and corresponding time, we have finally for friction of motion

$$f = \frac{G}{G_1} - \frac{G + G_1}{G_1} \cdot \frac{2s}{gt^2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If G, Fig. 59, is made just sufficient to start the block, or sledge, G_1, we have for the friction of rest

$$f = \frac{G}{G_1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

160. Results of Experiments on Sliding Friction.—Professor Thurston in his article on Friction (which the student will do well to read) in Johnson's Cyclopædia gives the following epitome of results from General Morin's experiments (made for the French Government in 1883):

TABLE FOR FRICTION OF MOTION.

No.	Surfaces.	Unguent.	Angle ϕ.	$f = \tan \phi$.
1	Wood on wood.	None.	14° to 26½°	0.25 to 0.50
2	Wood on wood.	Soap.	2° to 11½°	0.04 to 0.20
3	Metal on wood.	None.	26½° to 31½°	0.50 to 0.60
4	Metal on wood.	Water.	15° to 20°	0.25 to 0.35
5	Metal on wood.	Soap.	11½°	0.20
6	Leather on metal.	None.	29½°	0.56
7	Leather on metal.	Greased.	13°	0.23
8	Leather on metal.	Water.	20°	0.36
9	Leather on metal.	Oil.	8½°	0.15
10	Smoothest and best lubricated surfaces.	....	1¾° to 2°	0.03 to 0.036

For friction of rest, about 40% may be added to the coefficients in the above table.

In dealing with the stone blocks of an arch-ring, φ is commonly taken $= 30°$, i.e., $f = \tan 30° = 0.58$ as a low safe value; it is considered that if the direction of pressure between two stones makes an angle $> 30°$ with the normal to the joint (see § 161) slipping may take place (the adhesion of cement being neglected).

General Morin states that for a sledge on dry ground $f =$ about 0.66.

Weisbach gives for metal on metal, *dry* (R. R. brakes for example), $f =$ from 0.15 to 0.24. Trautwine's Pocket-Book gives values of f for numerous cases of friction.

161. Cone of Friction.—Fig. 170. Let A and B be two rough blocks, of which B is immovable, and P the resultant of all the forces acting on A, except the pressure from B. B can furnish any required normal pressure N to balance $P \cos \beta$, but the limit of its tangential resistance is fN. So long then as β is $< \varphi$ the angle of friction, or in other words, so long as the line of action of P is *within the "cone of friction"* generated by revolving OC about ON, the block A will not

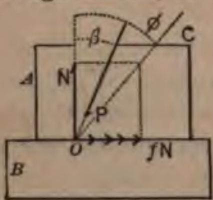


FIG. 170.

slip on B, and the tangential resistance of B is simply $P \sin \beta$; but if β is $> \varphi$, this tangential resistance being only fN and $< P \sin \beta$, A will begin to slip, with an acceleration.

162. Problems in Sliding Friction.—In the following problems f is supposed known at points where rubbing occurs, or is impending. As to the pressure N to which the friction is due, it is generally to be considered unknown until determined by the conditions of the problem. Sometimes it may be an advantage to deal with the single unknown force P (resultant of N and fN) acting in a line making the known angle φ with the normal (on the side *away* from the motion).

PROBLEM 1.—Required the value of the weight P, Fig. 171, the slightest addition to which will cause motion of the horizontal rod OB, resting on rough planes at 45°. The weight G of the rod may be applied at the middle. Consider the rod free; at each point of contact there is an unknown N and a friction due to it fN; the tension in the cord will be $= P$, since there is no acceleration and no friction at pulley. Notice the direction of the frictions, both opposing the impending motion. [The student should not rush to the conclusion that N and N_1 are equal, and are the same as would be produced by the components of G if the latter were transferred to A and resolved along AO and AB; but should await the legitimate results deduced by algebra, from the equations of condition for the equilibrium of a system of forces in a plane. Few problems in Mechanics are so simple as to admit of an immediate mental solution on inspection; and guess-work should be carefully avoided.]

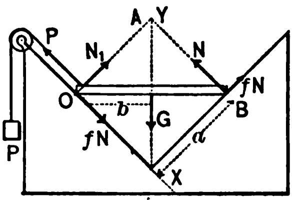


FIG. 171.

Taking an origin and two axes as in figure, we have (eqs. (2), § 36), denoting the sine of 45° by m,

$$\Sigma X \ldots \quad fN_1 + mG - N - P = 0; \quad . \quad . \quad (1)$$

$$\Sigma Y \ldots \quad N_1 + fN - mG = 0; \quad . \quad . \quad (2)$$

$$\Sigma(Pa) \ldots \quad fNa + Na - Gb = 0. \quad . \quad . \quad (3)$$

The three unknowns P, N, and N_1 can now be found. Divide (3) by a, remembering that $b : a = m$, and solve for N; substitute it in (2) and N_1 also becomes known; while P is then found from (1) and is

$$P = \frac{2mfG}{1+f} = \frac{f\sqrt{2}}{1+f} \cdot G.$$

PROBLEM 2.—Fig. 172. A rod, centre of gravity at middle, leans against a rough wall, and rests on an equally rough floor; how small may the angle α become before it slips? Let a = the half-length. The figure shows the rod free, and following the suggestion of § 162, a single unknown force P_1 making a known angle φ (whose tan $= f$) with the normal DE, is put in at D, leaning away from the direction of the impending motion, instead of an N and fN; similarly P_2 acts at C. The present system consisting of but three forces, the most direct method of finding α, without introducing the other two unknowns P_1 and P_2 at all, is to use the principle that if three forces balance, their lines of action must intersect in a point. That is, P_2 must intersect the vertical containing G, the weight, in the same point as P_1, viz., A.

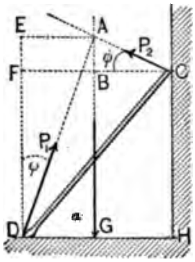


FIG. 172.

Now $\overline{EA}$, and also $\overline{BC}$, $= a \cos \alpha$,

$$\therefore ED = a \cos \alpha \cot \varphi \qquad \text{and} \qquad AB = a \cos \alpha \tan \varphi.$$

But DF, which $= 2a \sin \alpha$, $= DE - AB$;

$$\therefore 2a \sin \alpha = a \cos \alpha \, [\cot \varphi - \tan \varphi]. \quad . \quad . \quad (1)$$

Dividing by $\cos \alpha$, and noting that $\tan \varphi = f = 1 \div \cot \varphi$, we obtain for the required value of α

$$\tan \alpha = \frac{1}{2} \cdot \frac{1 - f^2}{f}; \quad \text{and finally,} \quad \tan \alpha = \cot 2\varphi,$$

after some trigonometrical reduction. That is, α is the complement of double the angle of friction.

Problem 3.—Fig. 173. Given the resistance Q, acting parallel to the fixed guide C, the angle α, and the (equal) coefficients of friction at the rubbing surfaces, required the

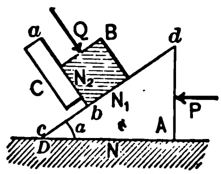


Fig. 173.

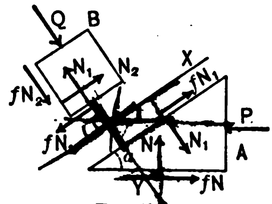


Fig. 174.

amount of the horizontal force P, at the head of the block A (or *wedge*), to overcome Q and the frictions. D is fixed, and ab is perpendicular to cd. Here we have four unknowns, viz., P, and the three pressures N, N_1, and N_2, between the blocks. Consider A and B as free bodies, separately (see Fig. 174), remembering Newton's law of action and reaction. The full values (e.g., fN) of the frictions are put in, since we suppose a slow uniform motion taking place.

For A, $\Sigma X = 0$ and $\Sigma Y = 0$ give

$$N_1 - N\cos\alpha + fN\sin\alpha - P\sin\alpha = 0; \quad . \quad . \quad (1)$$

$$fN_1 + N\sin\alpha + fN\cos\alpha - P\cos\alpha = 0. \quad . \quad . \quad (2)$$

For B, ΣX and ΣY give

$$Q - N_1 + fN_2 = 0; \ldots . (3) \quad \text{and} \quad N_2 - fN_1 = 0. \ldots (4)$$

Solve (4) for N_2 and substitute in (3), whence

$$N_1(1 - f^2) = Q. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Solve (2) for N, substitute the result in (1), as also the value of N_1 from (5), and the resulting equation contains but one unknown, P. Solving for P, putting for brevity

$$f\cos\alpha + \sin\alpha = m \quad \text{and} \quad \cos\alpha - f\sin\alpha = n,$$

we have $$P = \frac{(m + fn)Q}{(n \,.\, \cos\alpha + m \,.\, \sin\alpha)(1 - f^2)}. \quad . \quad . \quad (6)$$

Numerical Example of Problem 3.—If $Q = 120$ lbs., $f = 0.20$ (*an abstract number, and* $\therefore$ *the same in any system of units*), while $\alpha = 14°$, whose sine $= 0.240$ and cosine $= .970$, then

$$m = 0.2 \times .97 + 0.24 = 0.43 \quad \text{and} \quad n = .97 - .2 \times .24 = 0.92,$$

whence $$P = 0.64Q = 76.8 \text{ lbs.}$$

While the wedge moves 2 inches P does the work (or exerts an energy) of $2 \times 76.80 = 153.6$ in.-lbs. $= 12.8$ ft.-lbs.

For a distance of 2 inches described by the wedge horizontally, the block B (and $\therefore$ the resistance Q) has been moved through a distance $= 2 \times \sin 14° = 0.48$ in. along the guide C, and hence the work of $120 \times 0.48 = 57.6$ in.-lbs. has been done upon Q. Therefore for the supposed portion of the motion $153.6 - 57.6 = 96.0$ in.-lbs. of work has been lost in friction (converted into heat).

It is noticeable in eq. (6), that if f should $= 1.00$, $P = \propto$; and that if $\alpha = 90°$, $P = Q$, and there is no friction (the weights of the blocks have been neglected).

PROBLEM 4. *Numerical.*—With what minimum pressure P should the pulley A be held against B, which it drives by "frictional gearing," to transmit 2 H.P.; if $\alpha = 45°$, f for impending (relative) motion, i.e., for impending slipping $= 0.40$, and the velocity of the pulley-rim is 9 ft. per second?

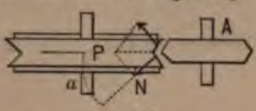


FIG. 173.

The limit-value of the tangential "grip"

$$T = \quad 2fN = 2 \times 0.40 \times P \sin 45°,$$
$$2 \text{ H. P.} = 2 \times 550 = 1100 \text{ ft.-lbs. per second.}$$

Putting $T \times 9$ ft. $= 1100$, we have

$$2 \times 0.40 \times \sqrt{\tfrac{1}{2}} \times P \times 9 = 1100 \,;\ \therefore P = 215 \text{ lbs.}$$

PROBLEM 6.—A block of weight G lies on a rough plane, inclined an angle β from the horizontal; find the pull P, making an angle α with the first plane, which will maintain a uniform motion *up* the plane.

Problem 7.—Same as 6, except that the pull P is to permit a uniform motion *down* the plane.

Problem 8.—The thrust of a screw-propeller is 15 tons. The ring against which it is exerted has a mean radius of 8 inches, the shaft makes one revolution per second, and $f = 0.06$. Required the H. P. lost in friction from this cause.

163. The Bent-Lever with Friction. Worn Bearing.—Fig. 176. Neglect the weight of the lever, and suppose the plumber-block so worn that there is contact along one element only of the shaft. Given the amount and line of action of the resistance R, and the line of action of P, required the amount of the latter for impending slipping in the direction of the dotted arrow. As P gradually increases, the shaft of the lever (or gear-wheel) rolls on its bearing until the line of contact has reached some position A, when rolling ceases and slipping begins. To find A, and the value of P, note that the total action of the bearing upon the lever is some force P_1, applied at A and making a known angle φ ($f = \tan\varphi$) with the normal AC. P_1 must be equal and opposite to the resultant of the known R and the unknown P, and hence graphically (a graphic is much simpler here than an analytical solution) if we describe about C a circle of radius $= r \sin\varphi$, r being the radius of shaft (or gudgeon), and draw a tangent to it from D, we determine DA as the line of action of P_1. If DG is made $= R$, to scale, and GF drawn parallel to $D \ldots P$, P is determined, being $= DE$, while $P_1 = DF$.

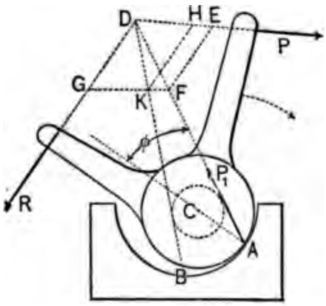


Fig. 176.

If the known force R is capable of acting as a working force, by drawing the other tangent DB from D to the "friction-circle," we have $P = DH$, and $P_1 = DK$, for impending rotation in an opposite direction.

If R and P are the tooth-pressures upon two spur-wheels, keyed upon the same shaft and nearly in the same plane, the

same constructions hold good, and for a continuous uniform motion, since the friction $= P_1 \sin \varphi$,

$$\left.\begin{array}{c}\text{the work lost in friction}\\ \text{per revolution,}\end{array}\right\} = [P_1 \sin \varphi] 2\pi r.$$

It is to be remarked, that without friction P_1 would pass through C, and that the moments of R and P would balance about C (for rest or uniform rotation); whereas with friction they balance about the proper tangent-point of the friction-circle.

Another way of stating this is as follows: So long as the resultant of P and R falls within the "dead-angle" BDA, motion is impossible in either direction.

If the weight of the lever is considered, the resultant of it and the force R can be substituted for the latter in the foregoing.

164. Bent-Lever with Friction. Triangular Bearing.—Like the preceding, the gudgeon is much exaggerated in the figure (177). For impending rotation in direction of the force P, the total actions at A_1 and A_2 must lie in known directions, making angles $= \varphi$ with the respective normals, and inclined away from the slipping. Join the intersections D and L. Since the resultant of P and R at D must act along DL to balance that of P_1 and P_2, having given one force, say R, we easily find $P = DE$, while P_1 and $P_2 = LM$ and LN respectively, LO having been made $= DF$, and the parallelogram completed.

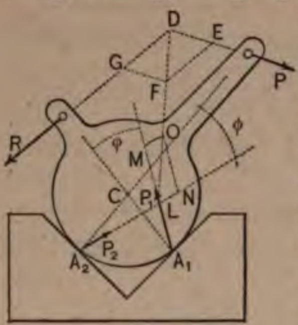


FIG. 177.

(If the direction of impending rotation is reversed, the change in the construction is obvious.) If $P_2 = 0$, the case reduces to that in Fig. 176; if the construction gives P_2 negative, the supposed contact at A_2 is not realized, and the angle A_2CA_1 should be increased, or shifted, until P_2 is positive.

As before, P and R may be the tooth-pressures on two

spur-wheels nearly in the same plane and on the same shaft; if so, then, for a uniform rotation,

Work lost in fric. per revol. $= [P_1 \sin \varphi + P_2 \sin \varphi] 2\pi r$.

165. Axle-Friction.—The two foregoing articles are introductory to the subject of axle-friction. When the bearing is new, or nearly so, the elements of the axle which are in contact with the bearing are infinite in number, thus giving an infinite number of unknown forces similar to P_1 and P_2 of the last paragraph, each making an angle φ with its normal. Refined theories as to the law of distribution of these pressures are of little use, considering the uncertainties as to the value of $f(= \tan \varphi)$; hence for practical purposes axle-friction may be written

$$F = f'R,$$

in which f' is a *coefficient of axle-friction* derivable from experiments with axles, and R the resultant pressure on the bearing. In some cases R may be partly due to the tightness of the bolts with which the cap of the bearing is fastened.

As before, the work lost in overcoming axle-friction *per revolution* is $= f'R2\pi r$, in which r is the radius of the axle. f', like f, is an abstract number. As in Fig. 176, a "friction-circle," of radius $= f'r$, may be considered as subtending the "dead-angle."

166. Experiments with Axle-Friction.—Prominent among recent experiments have been those of Professor Thurston (1872–73), who invented a special instrument for that purpose, shown (in principle only) in Fig. 178. By means of an internal spring, the amount of whose compression is read on a scale, a weighted bar or pendulum is caused to exert pressure on a projecting axle from which it is suspended. The axle is made to rotate at any desired velocity by some source of power, the axle-friction causing

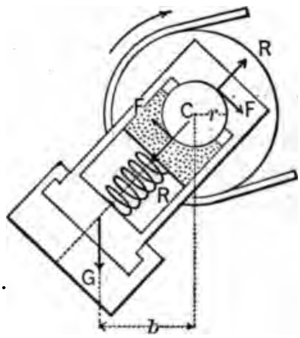


FIG. 178.

the pendulum to remain at rest at some angle of deviation from the vertical. The figure shows the pendulum free, the action of gravity upon it being G, that of the axle consisting of the two pressures, each $= R$, and of the two frictions (each being $F = f'R$), due to them. Taking moments about C, we have for equilibrium

$$2f'Rr = Gb,$$

in which all the quantities except f' are known or observed. The temperature of the bearing is also noted, with reference to its effect on the lubricant employed. Thus the instrument covers a wide range of relations.

General Morin's experiments as interpreted by Weisbach give the following practical results:

$$\left.\begin{array}{l}\text{For iron axles, in iron or}\\ \text{brass bearings}\end{array}\right\} f' = \left\{\begin{array}{l}0.054 \text{ for well-sustained lubrication;}\\ 0.07 \text{ to } .08 \text{ for ordinary lubrication.}\end{array}\right.$$

By "pressure per square inch on the bearing" is commonly meant the quotient of the total *pressure in lbs.* by the area in *square inches* obtained by multiplying the width of the axle by the length of bearing (this length is quite commonly four times the diameter); call it p, and the velocity of rubbing in *feet per minute*, v. Then, according to Rankine, to prevent overheating, we should have

$$p(v + 20) < 44800 \ldots \text{(not homog.)}.$$

Still, in marine-engine bearings pv alone often reaches 60,000, as also in some locomotives (Cotterill). Good practice keeps p within the limit of 800 (lbs. per sq. in.) for other metals than steel (Thurston), for which 1200 is sometimes allowed.

With $v = 200$ (feet per min.) Professor Thurston found that for ordinary lubricants p should not exceed values ranging from 30 to 75 (lbs. per sq. in.).

The product pv is obviously proportional to the power expended in wearing the rubbing surfaces, per unit of area.

167. Friction-Wheels.—A single example of their use will be given, with some approximations to avoid complexity. Fig. 179. G is the weight of a heavy wheel, P_1 is a known vertical resistance (tooth-pressure), and P an unknown vertical working force, whose value is to be determined to maintain a uniform rotation. The utility of the friction-wheels is also to be shown. The resultant of P_1, G, and P is a vertical force R, passing nearly through the centre C of the main axle which rolls on the four friction-wheels. R, resolved along CA and CB, produces (nearly) equal pressures, each being $N = R \div 2 \cos \alpha$, at the two axles of the friction-wheels, which rub against their fixed plumber-blocks. $R = P + P_1 + G_1$, and $\therefore$ contains the unknown P, but approximately $= G + 2P_1$, i.e., is nearly the same (in this case) whether friction-wheels are employed or not.

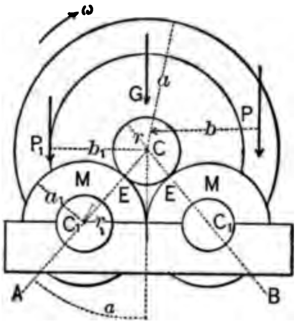


FIG. 179.

When G makes one revolution, the friction $f'N$ at each axle C_1 is overcome through a distance $= (r_1 : a_1)\, 2\pi r$, and

$$\left.\begin{array}{c}\text{Work lost per revol.}\\ \text{with}\\ \text{friction-wheels,}\end{array}\right\} = f'N\frac{r_1}{a_1}2\pi r = \frac{r_1}{a_1}\frac{1}{\cos\alpha}f'R2\pi r.$$

Whereas, if C revolved in a fixed bearing,

$$\left.\begin{array}{c}\text{Work lost per revol.}\\ \text{without}\\ \text{friction-wheels,}\end{array}\right\} = f'R2\pi r.$$

Apparently, then, there is a saving of work in the ratio $r_1 : a_1 \cos \alpha$, but strictly the R is not quite the same in the two cases; for with friction-wheels the force P is less than without, and R depends on P as well as on the known G and P_1. By diminishing the ratio $r_1 : a_1$, and the angle α, the saving is increased. If α were so large that $\cos \alpha < r_1 : a_1$, there would be no saving, but the reverse.

As to the value of P to maintain uniform rotation, we have

for equilibrium of moments about C, with friction-wheels (considering the large wheel and axle *free*),

$$Pb = P_1 b_1 + 2Tr, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which T is the tangential action, or "grip," between one pair of friction-wheels and the axle C which rolls upon them. T would not equal fN unless slipping took place or were impending at E, but is known by considering a pair of friction-wheels free, when $\Sigma (Pa)$ about C_1 gives

$$Ta_1 = f'Nr_1 = f'\frac{R}{2} \cdot \frac{r_1}{\cos \alpha},$$

which in (1) gives finally

$$P = \frac{b_1}{b}P_1 + \frac{r_1}{a_1 \cos \alpha}f'R\frac{r}{b}. \quad . \quad . \quad . \quad . \quad (2)$$

Without friction-wheels, we would have

$$P = \frac{b_1}{b}P_1 + f'R\frac{r}{b}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The last term in (2) is seen to be less than that in (3) (unless α is too large), in the same ratio as already found for the saving of work, supposing the R's equal.

If P_1 were on the same side of C as P, it would be of an opposite direction, and the pressure R would be diminished. Again, if P were horizontal, R would not be vertical, and the friction-wheel axles would not bear equal pressures. Since P depends on P_1, G, and *the frictions*, while the friction depends on R, and R on P_1, G, and P, an exact analysis is quite complex, and is not warranted by its practical utility.

Example.—If an empty vertical water-wheel weighs 25,000 lbs., required the force P to be applied at its circumference to maintain a uniform motion, with $a = 15$ ft., and $r = 5$ inches. Here $P_1 = 0$, and $R = G$ (nearly; neglecting the influence of P on R), i.e., $R = 25{,}000$ lbs.

First, without friction-wheels (adopting the foot-pound-second system of units), with $f' = .07$ (abstract number). From eq. (3) we have

$$P = 0 + 0.07 \times 25{,}000 \times (\tfrac{5}{12} \div 15) = 48.6 \text{ lbs.}$$

The work lost in friction per revolution is

$$fR2\pi r = 0.07 \times 25{,}000 \times 2 \times 3.14 \times \tfrac{5}{12} = 4580 \text{ ft.-lbs.}$$

Secondly, with friction-wheels, in which $r_1 : a_1 = \frac{1}{5}$ and $\cos \alpha = 0.80$ (i.e., $\alpha = 36°$). From eq. (2)

$$P = 0 + \tfrac{1}{5} \cdot \tfrac{10}{8} \times 48.6 = \text{only } 12.15 \text{ lbs.},$$

while the work lost per revolution

$$= \tfrac{1}{5} \cdot \tfrac{10}{8} \times 4580 = 1145 \text{ ft.-lbs.}$$

Of course with friction-wheels the wheel is not so steady as without.

In this example the force P has been simply enough to overcome friction. In case the wheel is in actual use, P is the weight of water actually in the buckets at any instant, and does the work of overcoming P_1, the resistance of the mill machinery, and also the friction. By placing P_1 pointing upward on the same side of C as P, and making b_1 nearly $= b$, R will $= G$ nearly, just as when the wheel is running empty; and the foregoing numerical results will still hold good for practical purposes.

168. Friction of Pivots.—In the case of a vertical shaft or axle, and sometimes in other cases, the extremity requires support against a thrust along the axis of the axle or pivot. If the end of the pivot is *flat* and also the surface against which it rubs, we may consider the pressure, and therefore the friction, as uniform over the surface. With a flat circular pivot, then, Fig. 180, the frictions on a small sector of the circle form a system of parallel forces whose resultant is equal to their sum, and is applied a distance of $\frac{2}{3}r$ from the centre. Hence the sum of the moments of all the frictions about the centre $= fR\frac{2}{3}r$, in which R is the axial pressure. Therefore a force P necessary to overcome the friction with uniform rotation must have a moment

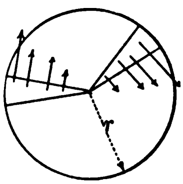


Fig. 180.

$$Pa = fR\tfrac{2}{3}r,$$

and the work lost in friction per revolution is

$$=fR2\pi \,.\, \tfrac{2}{3}r = \tfrac{4}{3}\pi fRr. \quad . \quad . \quad . \quad . \quad (1)$$

As the pivot and step become worn, the resultant frictions in the small sectors probably approach the centre; for the greatest wear occurs first near the outer edge, since there the product pv is greatest (see § 166). Hence for $\frac{2}{3}r$ we may more reasonably put $\frac{1}{2}r$.

Example.—A vertical flat-ended pivot presses its step with a force of 12 tons, is 6 inches in diameter, and makes 40 revolutions per minute. Required the H. P. absorbed by the friction. Supposing the pivot and step new, and f for good lubrication $= 0.07$, we have, from eq. (1) (*foot-lb.-second*),

Work lost per revolution

$$= .07 \times 24{,}000 \times 6.28 \times \tfrac{2}{3} \,.\, \tfrac{1}{4} = 1758.4 \text{ ft.-lbs.,}$$

and $\therefore$ work per second

$$= 1758.4 \times \tfrac{40}{60} \qquad\qquad = 1172.2 \text{ ft.-lbs.,}$$

which $\div$ 550 gives 2.13 H. P. absorbed in friction. If ordinary axle-friction also occurs its effect must be added.

If the flat-ended pivot is *hollow*, with radii r_1 and r_2, we may put $\frac{1}{2}(r_1 + r_2)$ instead of the $\frac{2}{3}r$ of the preceding.

It is obvious that the smaller the lever-arm given to the resultant friction in each sector of the rubbing surface the smaller the power lost in friction. Hence pivots should be made as small as possible, consistently with strength.

For a *conical pivot* and step, Fig. 181, the resultant friction in each sector of the conical bearing surface has a lever-arm $= \frac{2}{3}r_1$ about the axis A, and a value $>$ than for a flat-ended pivot; for, on account of the wedge-like action of the bodies, the pressure causing friction is greater. The sum of the moments of these resultant frictions about A is the same as if only two elements of the cone received pressure (each $= N = \frac{1}{2}R \div \sin\alpha$). Hence the

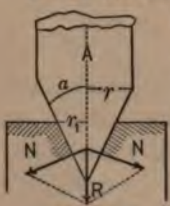


FIG. 181.

moment of friction of the pivot, i.e., the moment of the force necessary to maintain uniform rotation, is

$$Pa = f\,2N\frac{2}{3}r_1 = f\frac{R}{\sin\alpha}\,\frac{2}{3}r_1,$$

and work lost per revolution $\quad = \frac{4}{3}\pi f\frac{R}{\sin\alpha}r_1.$

By making r_1 small enough, these values may be made less than those for a flat-ended pivot of the same diameter.

In Schiele's "anti-friction" pivots the outline is designed according to the following theory for securing uniform vertical wear. Let p = the pressure per *horizontal* unit of area (i.e., $= R \div$ horizontal projection of the *actual rubbing surface*); this is assumed constant. Let the unit of area be small, for algebraic simplicity. The friction on the rubbing surface, whose horizontal projection = unity, is $= fN = f(p \div \sin\alpha)$ (see Fig. 182; the horizontal component of p is annulled by a corresponding one opposite). The work per revolution in producing wear on this area $= fN2\pi y$. But the *vertical depth* of wear per revolution is to be the same at all parts of the surface; and this implies that the same volume of material is worn away under each horizontal unit of area. Hence $fN2\pi y$, i.e., $f\frac{p}{\sin\alpha}2\pi y$, is to be constant for all values of y; and since fp and 2π are constant, we must have, as the law of the curve,

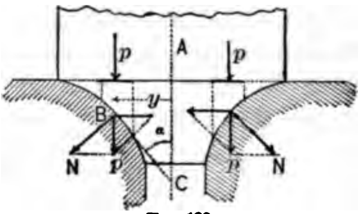


FIG. 182.

$$\frac{y}{\sin\alpha}, \text{ i.e., } \textit{the tangent } BC = \textit{the same at all points.}$$

This curve is called the "*tractrix*." Schiele's pivots give a very uniform wear at high speeds. The smoothness of wear prevents leakage in the case of cocks and faucets.

169. Normal Pressure of Belting.—When a perfectly flexible cord, or belt, is stretched over a smooth cylinder, both at rest,

the action between them is normal at every point. As to its amount, p, per linear unit of arc, the following will determine. Consider a semicircle of the cord free, neglecting its weight. Fig. 183. The force holding it in equilibrium are the tensions at the two ends (these are equal, manifestly, the cylinder being smooth; for they are the only two forces having moments about C, and each has the *same lever-arm*), and the normal pressures,

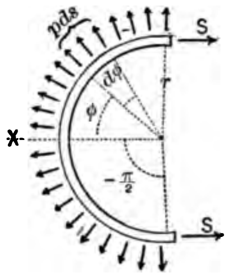


FIG. 183.

which are infinite in number, but have an intensity, p, per linear unit, which must be constant along the curve since S is the same at all points. The normal pressure on a single element, ds, of the cord is $= pds$, and its X component $= pds \cos\theta = prd\theta \cos\theta$. Hence $\Sigma X = 0$ gives

$$rp\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\cos\theta d\theta - 2S = 0, \text{ i.e., } rp\Big[_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\sin\theta = 2S;$$

$$\therefore rp[1 - (-1)] = 2S \quad \text{or} \quad p = \frac{S}{r}. \quad . \quad . \quad . \quad . \quad (1)$$

170. Belt on Rough Cylinder. Impending Slipping.—If friction is possible between the two bodies, the tension may vary along the arc of contact, so that p also varies, and consequently

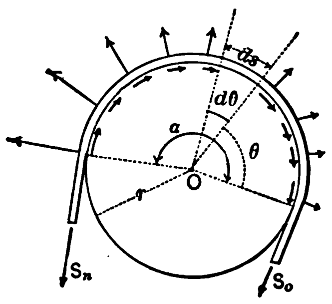


FIG. 184.

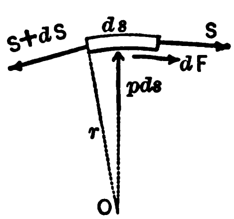


FIG. 185.

the friction on an element ds being $= fpds = f(S \div r)ds$, also varies. If *slipping is impending*, the law of variation of the tension S may be found, as follows: Fig. 184, in which the

impending slipping is toward the left, shows the cord free. For any element, ds, of the cord, we have, putting Σ (moms. about O) $= 0$ (Fig. 185),

$$(S + dS)r = Sr + dFr\,;\ \text{i.e.,}\ dF = dS,$$

or (see above) $dS = f(S \div r)ds$.

But $ds = rd\theta$; hence, after transforming,

$$fd\theta = \frac{dS}{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In (1) the two variables θ and S are separated; (1) is therefore ready for integration.

$$\therefore f\int_0^\alpha d\theta = \int^{S_n} \frac{dS}{S}\,;\ \text{i.e.,}$$

$$f\alpha = \log_e S_n - \log_e S_0 = \log_e\left[\frac{S_n}{S_0}\right]. \quad (2)$$

Or, by inversion, $\qquad S_0 e^{f\alpha} = S_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$

e, denoting the Naperian base, $= 2.71828+$; α of course is in π-measure.

Since S_n evidently increases very rapidly as α becomes larger, S_0 remaining the same, we have the explanation of the well-known fact that a comparatively small tension, S_0, exerted by a man, is able to prevent the slipping of a rope around a pile-head, when the further end is under the great tension S_n due to the stopping of a moving steamer. For example, with $f = \frac{1}{3}$, we have (Weisbach)

$$\begin{aligned}
\text{for } \alpha &= \tfrac{1}{4} \text{ turn, or } \alpha = \tfrac{1}{2}\pi,\ S_n = 1.69S_0\,;\\
&= \tfrac{1}{2} \text{ turn, or } \alpha = \pi,\ S_n = 2.85S_0\,;\\
&= 1 \text{ turn, or } \alpha = 2\pi,\ S_n = 8.12S_0\,;\\
&= 2 \text{ turns, or } \alpha = 4\pi,\ S_n = 65.94S_0\,;\\
&= 4 \text{ turns, or } \alpha = 8\pi,\ S_n = 4348.56S_0.
\end{aligned}$$

If slipping actually occurs, we must use a value of f for friction of motion.

Example.—A leather belt drives an iron pulley, covering one half the circumference. What is the limiting value of the

ratio of S_n (tension on driving-side) to S_0 (tension on following side) if the belt is not to slip, taking the low value of $f = 0.25$ for leather on iron?

We have given $f\alpha = 0.25 \times \pi = .7854$, which by eq. (2) is the Naperian log. of $(S_n : S_0)$ when slipping occurs. Hence the common log. of $(S_n : S_0) = 0.7854 \times 0.43429 = 0.34209$; i.e., if

$$(S_n : S_0) = 2.198, \text{ say } 2.2,$$

the belt will slip (for $f = 0.25$).

(0.43429 is the modulus of the common system of logarithms, and $= 1 : 2.30258$. See example in § 48.)

At very high speeds the relation $p = S \div r$ (in § 169) is not strictly true, since the tensions at the two ends of an element ds are partly employed in furnishing the necessary deviating force to keep the element of the cord in its circular path, the remainder producing normal pressure.

171. Transmission of Power by Belting or Wire Rope.—In the simple design in Fig. 186, it is required to find the *motive weight* G, necessary to overcome the given resistance R at a

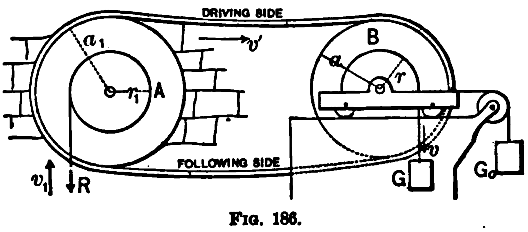


FIG. 186.

uniform velocity $= v_1$; also the proper stationary *tension weight* G_0 to prevent slipping of the belt on its pulleys, and the amount of power, L, transmitted.

In other words,

Given: $\left\{ \begin{array}{l} R, a, r, a_1, r_1;\ \alpha = \pi \text{ for both pulleys,} \\ v_1; \text{ and } f \text{ for both pulleys;} \end{array} \right\}$

Required: $\left\{ \begin{array}{l} L;\ G, \text{ to furnish } L;\ G_0 \text{ for no slip; } v \text{ the velocity} \\ \text{of } G;\ v' \text{ that of belt; and the tensions in belt.} \end{array} \right.$

Neglecting axle-friction and the rigidity of the belting, the power transmitted is that required to overcome R through a distance $= v_1$ every second, i.e.,

$$L = Rv_1. \quad \ldots \ldots \ldots \quad (1)$$

Since (if the belts do not slip)

$$a : r :: v' : v, \quad \text{and} \quad a_1 : r_1 :: v' : v_1,$$

we have $$v' = \frac{a_1}{r_1}v_1, \quad \text{and} \quad v = \frac{r}{a}\frac{a_1}{r_1}v_1. \quad \ldots \ldots \quad (2)$$

Neglecting the mass of the belt, and assuming that each pulley revolves on a gravity-axis, we obtain the following, by considering the free bodies in Fig. 187:

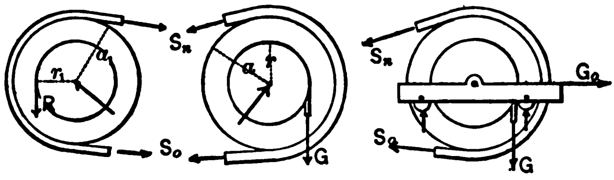


FIG. 187.

$$\Sigma\,(\text{moms.}) = 0 \text{ in } A \textit{ free} \text{ gives } Rr_1 = (S_n - S_0)a_1 ; \quad (3)$$

$$\Sigma\,(\text{moms.}) = 0 \text{ in } B \text{ free gives } Gr = (S_n - S_0)a ; \quad (4)$$

whence we readily find $$G = \frac{a}{r} \cdot \frac{r_1}{a_1}R.$$

Evidently R and G are inversely proportional to their velocities v_1 and v; see (2). This ought to be true, since in Fig. 186 G is the only working-force, R the only resistance, and the motions are uniform; hence (from eq. (XVI.), § 142)

$$Gv - Rv_1 = 0.$$

$\Sigma X = 0$, for B and truck free, gives

$$G_0 = S_n + S_0, \quad \ldots \ldots \ldots \quad (5)$$

while, for impending slip,

$$S_n = S_0 e^{f\pi}. \quad \ldots \ldots \ldots \quad (6)$$

By elimination between (4), (5), and (6), we obtain

$$G_0 = G\frac{r}{a} \cdot \frac{e^{f\pi}+1}{e^{f\pi}-1} = \frac{L}{v'} \cdot \frac{e^{f\pi}+1}{e^{f\pi}-1}, \quad . \quad . \quad . \quad (7)$$

and $$S_n = \frac{L}{v'} \cdot \frac{e^{f\pi}}{e^{f\pi}-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence G_0 and S_n vary directly as the power transmitted and inversely as the velocity of the belt. For safety G_0 should be made $>$ the above value in (7); corresponding values of the two tensions may then be found from (5), and from the relation (see § 150)

$$(S_n - S_0)v' = L. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

These *new* values of the tensions will be found to satisfy the condition of no slip, viz.,

$$(S_n : S_0) < e^{f\pi} (§ 170).$$

For leather on iron, $e^{f\pi} = 2.2$ (see example in § 170), as a low value. The belt should be made strong enough to withstand S_n safely.

As the belt is more tightly stretched, and hence elongated, on the driving than on the following side, it "*creeps*" backward on the driving and forward on the driven pulley, so that the former moves slightly faster than the latter. The loss of work due to this cause does not exceed 2 per cent with ordinary belting (Cotterill).

In the foregoing it is evident that the sum of the tensions in the two sides $= G_0$, i.e., = a constant, whether the power is being transmitted or not; and this is found to be true, both in theory and by experiment, when a tension-weight is not used, viz., when an initial tension S is produced in the whole belt before transmitting the power, then after turning on the latter the sum of the two tensions (driving and following) always $= 2S$, since one side elongates as much as the other contracts; it being understood that the pulley-axles preserve a constant distance apart.

172. Rolling Friction.—The few experiments which have been made to determine the resistance offered by a level road-

way to the uniform motion of a roller or wheel rolling upon it corroborate approximately the following theory. The word friction is hardly appropriate in this connection (except when the roadway is perfectly elastic, as will be seen), but is sanctioned by usage.

First, let the roadway or track be compressible, but *inelastic*, G the weight of the roller and its load, and P the horizontal force necessary to preserve a uniform motion (both of translation and rotation). The track (or roller itself) being compressed just in front, and not reacting symmetrically from behind, its resultant pressure against the roller is not at O vertically under the centre, but some small distance, $OD = b$, in front. (The successive crushing of small projecting particles has the same effect.) Since for this case of motion the forces have the same relations as if balanced (see § 124), we may put Σ moms. about $D = 0$,

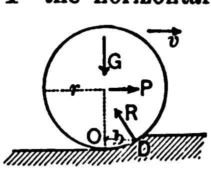


FIG. 188.

$$\therefore Pr = Gb; \quad \text{or,} \quad P = \frac{b}{r}G. \quad . \quad . \quad . \quad . \quad (1)$$

Coulomb found for

Rollers of lignum-vitæ on an oak track, $b = 0.0189$ inches;
Rollers of elm on an oak track, $b = 0.0320$ inches.

Weisbach's experiments give, for cast-iron wheels 20 inches in diameter on cast-iron rails,

$b = 0.0183$ inches;

and Rittinger, for the same, $b = 0.0193$ inches.

Pambour gives, for iron railroad wheels 39.4 inches in diameter,

$b = 0.0196$ to 0.0216 inches.

According to the foregoing theory, P, the "rolling friction" (see eq. (1)), is directly proportional to G, and inversely to the radius, if b is constant. The experiments of General Morin and others confirm this, while those of Dupuit, Poirée, and Sauvage indicate it to be proportional directly to G, and inversely to the square root of the radius.

Although b is a *distance* to be expressed in linear units, and not an abstract number like the f and f' for sliding and axle-friction, it is sometimes called a "coefficient of rolling friction." In eq. (1), b and r should be expressed in the same unit.

Of course if P is applied at the top of the roller its lever-arm about D is $2r$ instead of r, with a corresponding change in eq. (1).

With ordinary railroad cars the resistance due to axle and rolling frictions combined is about 8 lbs. per ton of weight on a level track. For wagons on macadamized roads $b = \frac{1}{2}$ inch, but on soft ground from 2 to 3 inches.

Secondly, when the roadway is *perfectly elastic*. This is chiefly of theoretic interest, since at first sight no force would be considered necessary to maintain a uniform rolling motion. But, as the material of the roadway is compressed under the roller its surface is first elongated and then recovers its former state; hence some rubbing and consequent sliding friction must

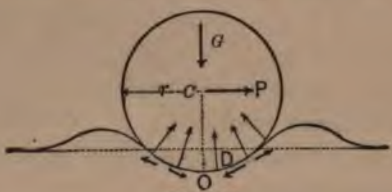


FIG. 189.

occur. Fig. 189 gives an exaggerated view of the circumstances, P being the horizontal force applied at the centre necessary to maintain a uniform motion. The roadway (rubber for instance) is heaped up both in front and behind the roller, O being the point of greatest pressure and elongation of the surface. The forces acting are G, P, the normal pressures, and the frictions due to them, and must form a balanced system. Hence, since G and P, and also the normal pressures, pass through C, the resultant of the frictions must also pass through C; therefore the frictions, or tangential actions, on the roller must be some forward and some backward

(and not all in one direction, as seems to be asserted on p. 260 of Cotterill's Applied Mechanics, where Professor Reynolds' explanation is cited). The resultant action of the roadway upon the roller acts, then, through some point D, a distance $OD = b$ ahead of O, and in the direction DC, and we have as before, with D as a centre of moments,

$$Pr = Gb, \quad \text{or} \quad P = \frac{b}{r}G.$$

If rolling friction is encountered *above as well as below* the rollers, Fig. 190, the student may easily prove, by considering three separate free bodies, that for uniform motion

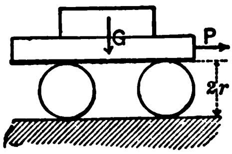


FIG. 190.

$$P = \frac{b + b_1}{2r}G, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where b and b_1 are the respective "coefficients of rolling friction" for the upper and lower contacts.

Example 1.—If it is found that a train of cars will move *uniformly* down an incline of 1 in 200, gravity being the only working force, and friction (both rolling and axle) the only resistance, required the coefficient, f', of axle-friction, the diameter of all the wheels being $2r = 30$ inches, that of the journals $2a = 3$ inches, taking $b = 0.02$ inch for the rolling friction. Let us use equation (XVI.) (§ 142), noting that while the train moves a distance s measured on the incline, its weight G does the work $G\frac{1}{200}s$, the rolling friction $\frac{b}{a}G$ (at the axles) has been overcome through the distance s, and the axle-friction (total) through the (relative) distance $\frac{a}{r}s$ in the journal boxes; whence, the change in kinetic energy being zero,

$$\frac{1}{200}Gs - \frac{b}{r}Gs - \frac{a}{r}f'Gs = 0.$$

Gs cancels out, the ratios $b:r$ and $a:r$ are $= \frac{1}{1500}$ and $\frac{1}{10}$, respectively (being ratios or abstract numbers they have the

same numerical values, whether the inch or foot is used), and solving, we have

$$f = 0.05 - 0.0066 = 0.043.$$

Example 2.—How many pounds of tractive effort per ton of load would the train in Example 1 require for uniform motion on a level track?

173. Railroad Brakes.—During the uniform motion of a railroad car the tangential action between the track and each wheel is small. Thus, in Example 1, just cited, if ten cars of eight wheels each make up the train, each car weighing 20 tons, the backward tangential action of the rails upon each wheel is only 25 lbs. When the brakes are applied to stop the train this action is much increased, and is the only agency by which the rails can retard the train, directly or indirectly: *directly*, when the pressure of the brakes is so great as to prevent the wheels from turning, thereby causing them to "skid" (i.e., slide) on the rails; *indirectly*, when the brake-pressure is of such a value as still to permit perfect rolling of the wheel, in which case the rubbing (and heating) occurs between the brake and wheel, and the tangential action of the rail has a value equal to or less than the friction of rest. In the first case, then (skidding), the retarding influence of the rails is the *friction of motion* between rail and wheel; in the second, a force which may be made as great as the *friction of rest* between rail and wheel. Hence, aside from the fact that skidding produces objectionable flat places on the wheel-tread, the brakes are more effective if so applied that skidding is *impending*, but not actually produced; for the friction of rest is usually greater than that of actual slipping (§ 160). This has been proved experimentally in England. The retarding effect of axle and rolling friction has been neglected in the above theory.

Example 1.—A twenty-ton car with an initial velocity of 80 feet per second (nearly a mile a minute) is to be stopped on a level within 1000 feet; required the necessary friction on each of the eight wheels.

Supposing the wheels not to skid, the friction will occur

between the brakes and wheels, and is overcome through the (relative) distance 1000 feet. Eq. (XVI.), § 142, gives (foot-lb.-second system)

$$0 - 8F \times 1000 = 0 - \frac{1}{2}\frac{40000}{32.2}(80)^2,$$

from which F (= friction at circumference of each wheel) = 496 lbs.

Example 2.—Suppose skidding to be impending in the foregoing, and the coefficient of friction of rest (i.e., impending slipping) between rail and wheel to be $f = 0.20$. In what distance will the car be stopped?

Example 3.—Suppose the car in Example 1 to be on an up-grade of 60 feet to the mile. (In applying eq. (XVI.) here, the weight 20 tons will enter as a resistance.)

Example 4.—In Example 3, consider all four resistances, viz., gravity, rolling friction, and brake and axle frictions, the distance being 1000 ft., and F the unknown quantity.

174. Estimation of Engine and Machinery Friction.—According to Professor Cotterill, a convenient way of estimating the work lost in friction in a steam-engine and machinery driven by it is the following:

Let p_m = mean effective steam-pressure per unit of area of piston, and conceive this composed of three portions, viz.,

p_0 = the necessary pressure to drive the engine alone unloaded, at the proper speed;

p'_m = pressure necessary to overcome the resistance caused by the useful work of the machines;

ep'_m = pressure necessary to overcome the friction of the machinery, and that of the engine over and above its friction when unloaded. This is about 15% of p'_m (i.e., $e = 0.15$), except in large engines, and then rather less.

That is, by formula, F being the piston-area and l the length of stroke, the work per stroke is thus distributed:

$$Fp_m l = F[(1 + e)p'_m + p_0]l,$$

p_0 is "from 1 to $1\frac{1}{2}$, or in marine engines 2 lbs. or more per square inch."

175. Anomalies in Friction.—Experiment has shown considerable deviation under certain circumstances from the laws of friction, as stated in § 157 for sliding friction. At pressures below $\frac{3}{4}$ lb. per sq. inch the coefficient f increases when the pressure decreases, while above 500 lbs. (Rennie, with iron and steel) it increases with the pressure. With high velocities, however, above 10 ft. per second, f is much smaller as the velocity increases (Bochet, 1858).

As for axle-friction, experiments instituted by the Society of Mechanical Engineers in England (see the London *Engineer* for March 7 and 21, 1884) gave values for f' less than $\frac{1}{100}$ when a "bath" of the lubricant was employed. These values diminished with increase of pressure, and increased with the velocity (see below, Hirn's statement).

Professor Cotterill says, "It cannot be doubted that for values of pv (see § 166) > 5000 the coefficient of friction of well-lubricated bearings of good construction diminishes with the pressure, and may be much less than the value at low speeds as determined by Morin" (p. 259 of his Applied Mechanics).

Professor Thurston's experiments confirmed those of Hirn as to the following relation: "The friction of lubricated surfaces is nearly proportional to the square root of the area and pressure." Hirn also maintained that, "in ordinary machinery, friction varies as the square root of the velocity."

176. Rigidity of Ropes.—If a rope or wire cable passes over a pulley or sheave, a force P is required on one side greater than the resistance Q on the other for uniform motion, aside from axle-friction. Since in a given time both P and Q describe the same space s, if P is $> Q$, then Ps is $> Qs$, i.e., the work done by P is $>$ than that expended upon Q. This is because some of the work Ps has been expended in bending the stiff rope or cable, and in overcoming friction between the strands, both where the rope passes upon and where it leaves

the pulley. With hemp ropes, Fig. 191, the material being nearly inelastic, the energy spent in bending it on at D is nearly all lost, and energy must also be spent in straightening

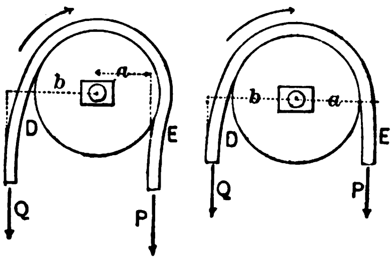


FIG. 191.

it at E; but with a wire rope or cable some of this energy is restored by the elasticity of the material. The energy spent in friction or rubbing of strands, however, is lost in both cases.

The figure shows geometrically why P must be $> Q$ for a uniform motion, for the lever-arm, a, of P is evidently $< b$ that of Q. If axle-friction is also considered, we must have

$$Pa = Qb + f'(P + Q)r,$$

r being the radius of the journal.

Experiments with cordage have been made by Prony, Coulomb, Eytelwein, and Weisbach, with considerable variation in the results and formulæ proposed. (See Coxe's translation of vol. i., Weisbach's Mechanics.)

With pulleys of large diameter the effect of rigidity is very slight. For instance, Weisbach gives an example of a pulley five feet in diameter, with which, Q being $= 1200$ lbs., $P = 1219$. A wire rope $\frac{3}{8}$ in. in diameter was used. Of this difference, 19 lbs., only 5 lbs. was due to rigidity, the remainder, 14 lbs., being caused by axle-friction. When a hemp-rope 1.6 inches in diameter was substituted for the wire one, $P - Q = 27$ lbs., of which 12 lbs. was due to the rigidity. Hence in one case the loss of work was less than $\frac{1}{2}$ of 1%, in the other about 1%, caused by the rigidity. For very small sheaves and thick ropes the loss is probably much greater.

177. Miscellaneous Examples.—*Example* 1. The end of a shaft 12 inches in diameter and making 50 revolutions per minute exerts against its bearing an axial pressure of 10 tons and a lateral pressure of 40 tons. With $f = f' = 0.05$, required the H. P. lost in friction.

Example 2.—A leather belt passes over a vertical pulley, covering half its circumference. One end is held by a spring balance, which reads 10 lbs. while the other end sustains a weight of 20 lbs., the pulley making 100 revolutions per minute. Required the coefficient of friction, and the H. P. spent in overcoming the friction. Also suppose the pulley turned in the other direction, the weight remaining the same.

Example 3.—A grindstone with a radius of gyration = 12 inches has been revolving at 120 revolutions per minute, and at a given instant is left to the influence of gravity and axle friction. The axles are $1\frac{1}{2}$ inches in diameter, and the wheel makes 160 revolutions in coming to rest. Required the coefficient of axle-friction.

Example 4.—A board A, weight 2 lbs., rests horizontally on another B; coefficient of friction of rest between them being $f = 0.30$. B is now moved horizontally with a uniformly accelerated motion, the acceleration being = 15 feet per "square second;" will A keep company with it, or not?